Math 414 Lecture 10

The original problem is called the primal problem.

For any constraint $s$, the value of its dual variable $s$ is the rate of change of the optimal value with respect to the constant of the constraint. Put simply

**Marginal Value Theorem.**

If $s$ is the dual variable of a constraint, then adding $\pm 1$ to its constant (other constraints permitting) increases the optimal value $z$ by $\pm s$.

The Marginal Value Theorem also determines the dual units. If the optimal value $z$ is dollars and the constraint constant $b$ is hours, then the dual variable $s$ is the rate of change of $z$ w.r.t $b = \Delta z/\Delta b$. The units are dollars/hour.

Basically, $s = \Delta z/\Delta b$ is the amount the optimal value increases when the constant in the constraint is increased by 1.

In economics, the dual variables are known as shadow prices (Linsolve’s word) or marginal values.

If you multiply a constraint by -1, you have to replace the dual variable $s$ by -$s$. Unlike the two-phase procedure, for duality, we avoid multiplying by -1.

**Find the duals of the problems below.**

<table>
<thead>
<tr>
<th><strong>Primal Problem</strong></th>
<th><strong>Dual Problem</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Max $z = C \cdot X$</td>
<td>Min $z = B \cdot W$</td>
</tr>
<tr>
<td>with $W: AX \leq B$</td>
<td>with $X: A^T W \geq C$</td>
</tr>
<tr>
<td>$X \geq 0$</td>
<td>$W \geq 0$.</td>
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<td>with $X: A^T W \geq C$</td>
</tr>
<tr>
<td>$X \geq 0$</td>
<td>$W$ unrestricted.</td>
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<th><strong>Primal Problem</strong></th>
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<tr>
<td>Max $z = 8x + 9y$</td>
<td>Min $z = 5r + 6s$</td>
</tr>
<tr>
<td>with $x \geq 0$, $y \geq 0$</td>
<td>with $r \geq 0$, $s \geq 0$.</td>
</tr>
<tr>
<td>$r: 2x - 3y \leq 5$</td>
<td>$x: 2r + 4s \geq 8$</td>
</tr>
<tr>
<td>$s: 4x + 8y \geq 6$</td>
<td>$y: -3r + 8s = 9$</td>
</tr>
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<table>
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<tr>
<th><strong>Negated Primal</strong></th>
<th><strong>Dual</strong></th>
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<tr>
<td>Max $z = 5r + 6s$</td>
<td>$w \geq 0$</td>
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Recall:
- Upward constraints $\rightarrow$ dual variables $\geq 0$,
- Downward constraints $\rightarrow$ dual variables $\leq 0$,
- equalities $\rightarrow$ unrestricted dual variables.

Recall: For a primal constraint $s$ with constant $b$, the dual variable $s = \frac{dz}{db}$ is the rate of change of $z$ w.r.t $b = \Delta z/\Delta b$ (other constraints permitting) the amount the optimal value $z$ increases when the constraint constant $b$ increases by 1.

**Solve geometrically and calculate the dual variables as rates of change.**

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<tr>
<th><strong>Primal</strong></th>
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<tr>
<td>Min $z = x + y$</td>
<td>Max $z$ $=$</td>
</tr>
<tr>
<td>$r: -x + y \leq 1$</td>
<td>$r =$</td>
</tr>
<tr>
<td>$s: x + y \geq 3$</td>
<td>$s =$</td>
</tr>
<tr>
<td>$t: x - y \leq 1$</td>
<td>$t =$</td>
</tr>
<tr>
<td>$x,y \geq 0$</td>
<td></td>
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</tbody>
</table>

Primal and dual problems have the same optimal value.

**Primal**
- Max $z = 5x$
- $w: x \leq 3$
- $x \geq 0$

**Dual**
- Min $z = 3w$
- $w: x \geq 5$
- $w \geq 0$

ANS. max $z = 15$ at $x = 3$

ANS. min $z = 15$ at $w = 5$

For any feasible solution $x$ of this primal problem and any feasible solution $w$ of the dual problem we have $z = 5x \leq 5 \cdot 3 = 15 = 3 \cdot 5 \leq 3w = z'$. Since $x \leq 3$ $5 \leq w$.

Thus $z \leq z'$.

In general, the optimal values $z$ and $z'$ of a primal and a dual problem are the same.

For nonoptimal values, the objective $z$ of the maximizing problem is $\leq$ the objective $z'$ of the minimizing problem.

**Negated Primal**
- Max $z = 5x$
- $w: x \geq 0$

**Dual**
- Min $z = -w$
- $w \leq 0$

(1) LPSolve doesn’t allow negated constraint variables -w.

(2) LPSolve does nonnegative and unrestricted (free) variables but not negative variables. For these, you have to take care of the sign.

To run LPSolve on a negative variable $s \leq 0$:
- Replace $s$ everywhere by -$s$
- Replace $s \leq 0$ by $s \geq 0$.
- Run LPSolve.
- Negate the answer to get back $s$.
Max $z = -8x - 9y$
with
\[x \quad y\]
\[r : \quad 2x - 2y = -5\]
\[s : \quad -x + 2y \geq 3\]
\[x \leq 0 \quad y \text{ unrestricted}\]
Change to
Max $z = 8(-x) - 9y = -8x - 9y$
with
\[-x \quad y\]
\[r : \quad -2x - 2y = -5\]
\[s : \quad x + 2y \geq 3\]
\[x \geq 0, \quad y \text{ unrestricted}\]
Run LPSolve with the line: free y;
LPSolve gives: $x = 2, \quad y = .5$
Negate x to get the answer for the original problem:
Answer: $x = -2, \quad y = .5$

**Duality Theorem** (Gale, Kuhn, Tucker). For any primal problem and its dual:

(a) An optimal value for one problem is also an optimal value for the other. (In economics, maximizing profits and minimizing costs are dual but equivalent objectives.)

(b) The objective values of feasible solutions of the maximizing problem are $\leq$ those of the minimizing problem.

(c) If both problems have feasible solutions, then both problems have optimal solutions.

(d) If a feasible primal solution has the same objective value as a feasible dual solution then both are optimal.

(e) One has no feasible solutions iff the other is unbounded in the gradient direction.

**Proof of (a)** Roughly speaking (there are sign changes and missing/added variables), an optimal tableau for the primal problem transposes to an optimal tableau for the dual problem. Since the optimal value lies in the bottom right corner, it transposes to itself. Hence it is the same for both problems.

**Proof of (b)** Suppose the optimal objective value for both problems is $z$. Let $z_\delta$ be some nonoptimal value for the maximizing problem; let $z_\sigma$ be some nonoptimal value for the minimizing problem. Then $z_\delta \leq z$ since $z$ is the maximum value for the maximizing problem and $z \leq z_\sigma$ since $z$ is the minimum value for the minimizing problem. Thus $z_\delta \leq z \leq z_\sigma$ which implies $z_\delta \leq z_\sigma$.

**Proof of (c)** If the minimizing problem has a feasible solution, then its objective value is $\geq$ that of all maximizing solutions. Hence the nonempty set of maximizing solutions is bounded in the gradient direction and hence has an optimum value.

**Proof of (d)** Suppose both maximizing and minimizing problems have a common objective value $z$. By part (b) this value is $\geq$ to all values of the maximizing problem. Hence it is the optimal value for the maximizing problem. Similarly it is optimal value for the minimizing problem.

**Proof of (e)** Draw a picture. \(\square\)

Henceforth, we will use the same optimal variable, say $z$, for both primal and dual problems.