Math 414  Lecture 16

Constant column sensitivity

Identify a “solution” with the constraint boundaries on which it lies (the constraints with 0 slacks). Thus a solution will move when its constraint lines move.

Let \( b_r \) = the right-hand constant of constraint \( r \).

When this constant changes, the constraint boundary moves. So does any optimal solution on the boundary. It remains optimal but may become infeasible.

Let \( B(b_r \rightarrow p) \) be the original constant column \( B \) with \( b_r \) replaced by a new variable \( p \). By the Constant Column Theorem, the final constant column \( B' = T \cdot B(b_r \rightarrow p) \).

This expresses the final solution as a function of \( p \). It shows how the solution changes with respect to changes in the constant \( b_r \).

The final solution is feasible iff the constant column is \( \geq 0 \) iff \( B' \geq 0 \). Thus the range of values of \( b_r \) for which the solution is feasible is the set of \( p \) such that \( B' \geq 0 \).

■ Primal problem
max \( z = y \)

with
\[
\begin{align*}
r: & -x + y \leq 1 \leftarrow b_r \\
s: & x + y \leq 3 \leftarrow b_s \\
t: & x - y \leq 1 \leftarrow b_t \\
x, & y \geq 0
\end{align*}
\]

■ What happens when \( b_s = 3 \) changes?

Initial matrix

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>r</th>
<th>s</th>
<th>t</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>r</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>s</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>t</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Complete the final tableau.

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>r</th>
<th>s</th>
<th>t</th>
<th>b'</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-0.5</td>
<td>0.5</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Thus \( T \) is the shaded matrix in the final tableau and \( B(b_s \rightarrow p) = [1; p; 1], \ B' = [.5 + .5p; -.5 + .5p; 2] \).

This is feasible
\[
\begin{align*}
\text{iff} & \ .5 + .5p \geq 0 \ \text{and} \ -.5 + .5p \geq 0 \ \text{and} \ 2 \geq 0 \\
\text{iff} & \ 1 + p \geq 0 \ \text{and} \ -1 + p \geq 0 \\
\text{iff} & \ p \geq -1 \ \text{and} \ p \geq 1 \\
\text{iff} & \ p \geq 1 \ \text{iff} \ p \in [1, \infty).
\end{align*}
\]

Write the answer as an interval.

Use the Objective Coefficient Theorem to get the objective value at the bottom right corner.

Answer. If \( b_s = 3 \) is replaced by a variable \( p \), the optimal solution as a function of \( b_s = p \) is
\[
\begin{align*}
\text{max} & \ z = .5 + .5p \\
\text{at} & \ x = -.5 + .5p \\
y & = .5 + .5p \ \text{for} \ b_s = p \in [1, \infty).
\end{align*}
\]

■ Is the solution still feasible if \( b_s = 3 \) changes to \( b_s = 0 \)?

■ What is the optimal solution if \( b_s = 3 \) is changed to 6?

Objective coefficient sensitivity

DEFINITION. For any primal variable \( x \), let \( c_x \) be the coefficient of \( x \) in the objective function.

■ If the objective is
\[
\min w = 3x - 4y - 2z,
\]

then \( c_x = 3, c_y = -4, c_z = -2 \).

If one of these coefficients, say \( c_x \) changes, what happens to the optimal solution?

The objective gradient vector changes but the constraints do not. Thus the feasible region is unchanged. Feasible and extreme solutions do not move and do not become unfeasible. However optimal solutions may cease to be optimal.

As the objective function rotates, the optimal solution may move from one extreme to another. At the point of changeover, the line segment between the extremes becomes optimal and the objective vector is perpendicular to the line segment. Mark these changeover directions, by drawing an outward pointing perpendicular vector from each line segment. Translate these boundary-segment perpendiculars to the origin.

■ For each extreme, draw lines to indicate the pie-shaped sector of objective coefficient vectors for which it is optimal.

Primal problem.
max \( z = 1x + 2y \)

with
\[
\begin{align*}
r: & y \leq 2 \\
s: & x + y \leq 3 \\
x, & y \geq 0
\end{align*}
\]
Then $c_x = 1$, $c_y = 2$ and the final tableau is

\[
\begin{array}{cccccc}
 & x & y & r & s & b \\
\hline
y & 0 & 1 & 1 & 0 & 2 \\
x & 1 & 0 & -1 & 1 & 1 \\
z & 0 & 0 & 1 & 1 & 5 \\
\end{array}
\]

Optimal solution. max $z = 5$ at $x = 1$, $y = 2$

For what values of $c_x$ is this solution optimal?
To determine this range of values, replace the
coefficient with a variable, say $p$, and recalculate the
objective row with the Objective Row Theorem.
The solution is optimal for the set of $p$ such that the
objective row coefficients are $\geq 0$, (exclude the unshaded
objective value entry).

\[
\begin{array}{cccccc}
 & x & y & r & s & b \\
\hline
p & 2 & 0 & 0 & 0 & 0 \\
p & x & 1 & 0 & -1 & 1 \\
z & 0 & 0 & 2-p & p & p+4 \\
\end{array}
\]

The solution is optimal
iff $2-p \geq 0$ and $p \geq 0$
iff $2 \geq p$ and $p \geq 0$
iff $p \in [0, 2]$.

Answer. The range is $c_y = p \in [0, 2]$.

Answer. If $c_x = 1$ is replaced by $p$, the solution $(x, y) = (1, 2)$ is optimal for
max $z = px + 2y = p1 + 2 \cdot 2 = p + 4$
when $c_x = p \in [0, 2]$.