Math 414 Lecture 20
Exam 3 covers Lectures 15 - 21

Transportation tableaus
- Bottom row: the values \( w_1, w_2, \ldots \) of the dual demand variables.
- Rightmost column: the values \( p_1, p_2, \ldots \) of the dual supply variables.
- The square for \( x_{ij}/o_{ij} \) has \( c_{ij} \) in the upper left corner.
- If \( x_{ij} \) is a basic variable, the square contains the value of \( x_{ij} \) which is circled. By complementary slackness, \( o_{ij} = 0 \).
- If \( x_{ij} \) is a parameter (in this case, \( x_{ij} = 0 \)), the square contains its objective row entry \( o_{ij} = c_{ij} - p_i - w_j \).

Transportation tableaus = Simplex tableaus − interior coefficients.

The greedy algorithm gets initial transportation matrix.

<table>
<thead>
<tr>
<th>( p_i )</th>
<th>( W_1 )</th>
<th>( W_2 )</th>
<th>( W_3 )</th>
<th>( W_4 )</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>8</td>
<td>6</td>
<td>1</td>
<td>1</td>
<td>50</td>
</tr>
<tr>
<td>( P_3 )</td>
<td>6</td>
<td>0</td>
<td>20</td>
<td>3</td>
<td>50</td>
</tr>
<tr>
<td>Demand</td>
<td>30</td>
<td>20</td>
<td>40</td>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>

We successively improve this to an optimal solution. The \( x_{ij} \)'s with circled values are the initial basic variables. Copy them to the initial transportation tableau and circle them. The \( i^{th} \) dual constraint with slacks is \( x_{ij} = p_i + w_j + o_{ij} = c_{ij} \).

**Loop:** For circled basic variables \( x_{ij} \), the dual slack is \( o_{ij} = 0 \) (Complementary Slackness Theorem). Thus \( p_i + w_j + o_{ij} = c_{ij} \) becomes \( p_i + w_j = c_{ij} \) (the basic equations).

There are 7 dual variables but only 6 independent equations \( p_i + w_j = c_{ij} \), hence only 6 are basic variables and the other, always pick \( p_1 \), is a parameter.

Thus \( p_1 = 0 \).

Use the 6 basic equations \( p_1 + w_j = c_{ij} \) to solve for the remaining 6 dual variables \( p_2, p_3, \ldots, w_4 \) in the supply column and demand row.

<table>
<thead>
<tr>
<th>( w_j )</th>
<th>( W_1 )</th>
<th>( W_2 )</th>
<th>( W_3 )</th>
<th>( W_4 )</th>
<th>( p_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>1</td>
<td>4</td>
<td>-3</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>8</td>
<td>6</td>
<td>1</td>
<td>40</td>
<td>1</td>
</tr>
<tr>
<td>( P_3 )</td>
<td>6</td>
<td>5</td>
<td>20</td>
<td>3</td>
<td>30</td>
</tr>
</tbody>
</table>

For uncircled squares, \( x_{ij} \) is a parameter with \( x_{ij} = 0 \).
Instead of writing \( x_{ij} \), write \( o_{ij} \). Since \( p_i + w_j + o_{ij} = c_{ij} \), we have \( o_{ij} = c_{ij} - p_i - w_j \). These \( o_{ij} \)'s are the dual slacks which are the simplex objective row entries. Write them in the uncircled parameter squares. This is now a transportation tableau.

If no objective row entry (uncircled interior square) \( o_{ij} \) is negative, stop, you have an optimal solution. Otherwise

- Choose the next entering basic variable as usual:
  - Pick the uncircled square with the most negative objective row entry \( o_{ij} \).
  - Erase \( o_{ij} \); shade its square; add an empty circle.
  - In row \( i \), the circled \( x_{ij} \) values must sum to \( s_i : \Sigma_j x_{ij} = s_i \).
  - In column \( j \), the circled \( x_{ij} \) values must sum to \( d_j \).
  - Thus if one circled \( x_{ij} \) value in a row or column increases (decreases), some other value must decrease (increase).
  - Chase these increases and decreases around until you find a loop which begins and ends at the shaded entering variable. Its other corners are circled basic values which must increase or decrease.
  - Alternately mark the corners of the loop “+” or “−” starting with a “+” on the shaded entering variable. As the shaded entering basic value increases, the “+” values must increase and the “−” values must decrease.
  - How much can be added and subtracted from the loop variables? The “−” values cannot decrease below 0 since \( x_{ij} \geq 0 \). The maximum amount of change = the minimum of the values labeled “−”.
  - Add this to the “+” variables; subtract it from the “−” variables.
  - The “−” variable with the minumum value goes to 0. This is the departing basic variable (if two or more go to 0, pick one to be the new parameter, the other remains a degenerate basic). Remove the square’s circled basic variable.
- End of loop. Repeat this loop (recalculate all \( p_i \)'s, \( w_j \)'s, and \( o_{ij} \)'s). Stop when all uncircled values \( o_{ij} \) are \( \geq 0 \).

Picking the most negative \( o_{ij} \) equals selecting a simplex column with the most negative objective coefficient.

Picking the first basic variable which goes to 0 equals picking the variable with the minimum \( \theta \) ratio to be the departing variable.

**Transportation algorithm**

Start with an initial transportation matrix (greedy algorithm). Repeat until an optimal solution is found:

- Set \( p_1 = 0 \).
- For the remaining \( p_i \), \( w_j \), calculate \( p_i, w_j \) using \( p_i + w_j = c_{ij} \) for each circled basic \( x_{ij} \).
- Calculate \( o_{ij} \) using \( o_{ij} = c_{ij} - p_i - w_j \) for each parameter \( x_{ij} \).
- If no objective coefficient (uncircled \( o_{ij} \)) is negative, stop. You have an optimal solution. Otherwise:
  - Pick the variable with the most negative \( o_{ij} \) as the entering variable. Erase \( o_{ij} \); shade the square; add a circle.
  - Find a loop, starting with the entering variable and whose other corners are basic (circled). Alternately mark the corners “+” and “−”, starting with a “+” on the entering variable.
  - Increase the “+” and decrease the “−” corner by the minimum of the “−” values.
  - Remove the circled departing variable (a minimum “−” value) which went to 0.
This is the final tableau since no objective row coefficient (uncircled $o_{ij}$) is negative.

Solution:
\[
\min z = 1 \cdot 30 + 4 \cdot 10 + 1 \cdot 50 + 7 \cdot 10 + 1 \cdot 40 + 3 \cdot 0 = 230
\]
at \begin{align*}
x_{11} &= 30, & x_{12} &= 10, & x_{24} &= 50, & x_{32} &= 10, & x_{33} &= 40, & x_{34} &= 0, \\
\text{the rest} &= 0
\end{align*}