Two-person zero-sum games

A game is played for money between two players: player I and player II. It is a 0-sum game if I’s winnings + II’s winnings = 0, i.e., what I wins, II loses. Each player has a finite number of ways or strategies in which he can play the game. Each time the game is played, the amount I wins = the amount player II loses = the payoff. The payoff matrix lists I’s strategies in the left column and II’s strategies in the top row; the entries in the matrix are the payoffs (I’s wins, II’s losses) which result when I and II play according to their strategies. Since the amount that I wins = the negative of the amount II wins, the sum of their amounts is 0. Thus these games are called zero-sum games.

Suppose I and II choose numbers from \{a, b, c\}. If the letters equal, both get $0. Otherwise the one with the alphabetically last letter gets $1, the other loses $1. Thus if I chooses b and II chooses c, I loses $1 with payoff -1.

When the payoff is higher, I wins more, \(\cdot\) I tries to maximize the payoff (his win).
When the payoff is lower, II wins more, \(\cdot\) II tries to minimize the payoff (his loss).

The payoff matrix is:

<table>
<thead>
<tr>
<th>II</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>Expected win for I</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

If player I consistently chooses a and II plays intelligently, what can I expect to win? Likewise if I consistently chooses b or c.

Now suppose the matrix is

<table>
<thead>
<tr>
<th>II</th>
<th>x</th>
<th>y</th>
<th>Expected win for I</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>9</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>8</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

I has 2 strategies a, b; II has 2 strategies x, y. If I consistently plays strategy a or b what can he expect to win?

**Lemma.** The amount I can expect to win by consistently playing the strategy for a given row is the minimum payoff for that row.

**Theorem.** The strategy for I with the maximum expected win is the strategy whose row has the maximum minimum. This is the maximin strategy for I.

For I, playing the maximin strategy is better than playing any other single strategy but playing a random mixed strategy may have a higher average payoff.

\[
\begin{array}{c|c|c|c}
\text{II} & H & T & \text{Expected win for I} \\
\hline
H & 1 & -1 & \\
T & -1 & 1 & \\
\hline
.5H+.5T & & & \\
\end{array}
\]

If I consistently plays H or T, he can expect to lose $1.

Suppose I plays according to the following mixed strategy: he chooses H and T randomly, each with probability 1/2. If II chooses H, then 1/2 of the time I wins $1 and 1/2 of time I loses $1. Thus on average I wins \(.5(1) + .5(-1) = 0\). Likewise, if II chooses T, I can expect to win $0. This mixed strategy for I gives him $0 on average; either strategy alone gives him $1.

**Lemma.** If \(f\) and \(g\) are intersecting functions, one increasing, one decreasing, then \(p\) maximizes the minimum of \(f(p)\) and \(g(p)\) iff \(f(p)=g(p)\) iff \(p\) minimizes the maximum of \(f(p)\) and \(g(p)\).

\[
\begin{align*}
\max z &= y \\
\text{with} \quad y &\leq f(p) \\
&\leq g(p)
\end{align*}
\]

Suppose player I chooses strategy \(A\) with probability \(p\) and chooses strategy \(B\) with probability \(1-p\).

If II plays strategy \(X\), then he loses $6 with probability \(p\) and loses $2 with probability \(1-p\). His average loss is \(p \cdot 6 + (1-p) \cdot 2 = 4p+2\).

If II plays \(Y\), his average loss is \(p \cdot 4 + (1-p) \cdot 8 = 8-4p\).

To minimize his losses, II selects the minimum of \(4p+2\) and \(8-4p\).

Thus to maximize his winnings, I should choose the \(p\) which maximizes the minimum of \(4p+2\) and \(8-4p\). By the Lemma above, this is the \(p\) such that \(4p+2 = 8-4p\). Thus \(8p = 6\) and hence \(p = 3/4\).

Answer. Player I should randomly choose strategy \(A\) with probability \(p = 3/4\) and choose strategy \(B\) with probability \(1/4\). His expected winnings are \(\min(4p+2, 8-4p)\), by choice of \(p = 4p+2 = 4(3/4)+2 = 5\).

Note, $5 is more than the $4 or $2 he gets by playing either A or B alone.