Math 414  Review 2

Material.
Lectures 8-14.
All exams are closed book: no notes, no text.
Bring a calculator, laptop and scratch paper.
Do not turn in your scratch paper; put your answers and all work you wish to show on the exam paper. Do the same with the practice exam.
You may only use SciLab and LPSolve on your laptop along with a calculator.

Definitions.
Degenerate basic solution, degenerate variable. Slack variables, extra variables. Phase one problem.
Primal and Dual problems, loosens, tightens, improves, worsens, upward, downward.
Dual variables, marginal value.

Be able to
Solve problems using the two phase method.
Write the dual of a linear programming problem.
Calculate dual variables by finding the rates of change.
Give economic interpretations of slack and dual variables.
Solve economic word problems.
Determine when a problem has a unique optimal solution and if not, find a second optimal solution.
Solve problems with secondary objectives.

Suggested Exercises. All homework exercises plus Page Problem
130:  1, 3, 5.
150:  1a, 3a, 5, 9, 21.
165:  1, 3, 5, 8, 9, 10.
183:  8, 9, 11.
202:  1, 3, 7, 9, 11.

Theorems. Be able to state the theorems below and answer questions about them. Proofs will not be required.

Lemma. The phase one problem has an optimal solution whose value is \( \neq 0 \) iff the original problem has no feasible solutions.

Lemma. Tightening/loosening a constraint worsens/improves the optimal value if it affects it at all.

Marginal Value Theorem. If \( s \) is the dual variable of a constraint, then adding \( \pm 1 \) to the constraint’s constant increases (other constraints permitting) the optimal value by \( \pm s \).

Duality Theorem (Gale, Kuhn, Tucker). For any primal problem and its dual:
(a) An optimal value for one problem is also an optimal value for the other.
(b) The objective values of feasible solutions of the maximizing problem are \( \leq \) those of the minimizing problem.
(c) If both problems have feasible solutions, then both problems have optimal solutions.
(d) If a feasible primal solution has the same objective value as a feasible dual solution then both are optimal.
(e) One has no feasible solutions iff the other is unbounded in the gradient direction.

Double Duality Theorem. The dual of a dual is the original primal problem.

Complementary Slackness Theorem. For any constraint with a slack variable and a dual variable: At least one of the two variables is a parameter. Hence at least one of the two is 0.

Lemma. For each feasible solution \( X \) for the primal there is a solution \( W \) (not necessarily feasible) for the dual which has the same objective value, i.e., \( C^T X = B^T W \).

Given. A canonical tableau with extra variables for a primal problem with constant column \( B \). Let \( X \) be the tableau’s solution. Let \( W \) be the associated solution for the dual problem. The cells of the final coefficient columns form a matrix \( T \) which was initially an identity matrix.

Write the original objective function coefficients (not their negatives) at the top of the tableau.
Label the rows with their basic variables.
To the left of these basic variables, write their objective coefficients (just copy them from the top objective row).
Let \( C_B \) be this column of objective coefficients.
Let \( t_j \) be the \( j \)th column of the tableau.
Let \( c_j \) be the objective coefficient at the top.
Let \( z_j = C_B t_j \).

Objective Row Theorem.
(a) The \( j \)th entry in the objective row is \( z_j - c_j = C_B t_j - c_j \).
(b) If the \( j \)th column is associated with a variable \( x \) of the primal problem, then \( z_j \) = the left side of the dual constraint \( x : a_{1j} w_1 + a_{3j} w_2 + \ldots + a_{mj} w_m \geq c_j \).
Thus the \( j \)th objective row entry = \( z_j - c_j \) = the slack in the dual constraint for \( x \).
(c) If the \( j \)th column is associated with an initial extra or slack variable \( w \) for a primal constraint then \( c_j = 0 \). Hence the objective row entry \( z_j - c_j \) is \( z_j \).
For the dual variable \( w, w = \) the objective row entry \( z_j \), or \( w = -z_j \). The sign is determined by the restriction \( w \geq 0 \), or \( w \leq 0 \). If \( w \) is unrestricted, the Marginal Value Theorem determines the sign. Alternatively, you can systematically keep track of sign changes (multiplying a constraint by \(-1\) changes the sign of the dual variable).

Constant Column Theorem. The constant column = \( T^T B \).
LEMMA. For standard primal problems, the $j$th dual constraint is satisfied iff $z_j \geq c_j$ iff $z_j - c_j \geq 0$.

COROLLARY. For standard primal problems, the $j$th dual constraint is satisfied iff $z_j \geq c_j$ iff $z_j - c_j \geq 0$.

PRIMAL-DUAL THEOREM.
(a) For every feasible basic solution of a primal problem, there is an associated solution for the dual problem with the same objective value.
(b) The primal solution is optimal iff the dual solution is optimal iff the dual solution is feasible.
(c) The primal solution is suboptimal iff the dual solution is superoptimal iff the dual solution is not feasible.

FACTORS WHICH CHANGE A DUAL VARIABLE’S SIGN:
- Changing max to min.
- Multiplying a primal constraint by -1.
- Subtracting a slack instead of adding.

THEOREM. If the objective coefficient of a final-tableau parameter is 0, select it as an entering variable to get a possibly different optimal basic solution (to be genuinely different some value must change, not just the set of basic variables). If all parameters have positive objective row coefficients, the optimal solution is unique.