**Math 416 Lecture 14**

**Poisson / G(x) / 1 Queues**

Suppose the arrivals are a Poisson process with arrival rate $\lambda$ and expected time $1/\lambda$ between arrivals and with $N_t$ the number of arrivals at time $t$.

Suppose the service times are mutually independent and have a cumulative distribution $G(x)$ and density function $g(x)$ and expected service time $1/\mu$.

Suppose there is one server.

The queue length $X_t$ at time $t$ is the number of people waiting plus the person, if any, being served. For each $j$, we want the long-term probability $\pi_j = \lim_{n \to \infty} P[X_n = j]$ that the queue length is $j$. The $p^j$ in Lecture 13 is $\pi_j$ here. Let $\pi_{\geq 1}$ be the long-term probability of having 1 or more in the queue.

Let $\rho = \lambda/\mu$. If $\rho > 1$, then $\rho > 1$ and the queue will grow forever. Hence we will assume $\rho < 1$. In this case, under rather general regularity conditions, the limit probabilities $\pi_n$ exist.

**Lemma.** If $\rho < 1$, $\pi_{\geq 1} = \lambda/\mu = \rho$, $\pi_0 = 1 - \rho$. \(\square\)

We now calculate $\pi_1$, $\pi_2$, ...

Let $Y_n$ be the number of people in the queue (including the one being served) just after the $n^{th}$ departure = the number of people waiting when the $n^{th}$ person has been served. $Y_n$ has both a discrete state space, $j \in \{0, 1, 2, \ldots\}$, and a discrete time set $n \in \{0, 1, 2, \ldots\}$. Since $Y_n$ is a subsequence of $X_n$, $\pi_j = \lim_{n \to \infty} P[X_n = j] = \lim_{n \to \infty} P[Y_n = j]$.

Since the arrivals are exponential and the service times independent, the queue is memoryless — the time to the next arrival and the current customer’s service time do not depend on the past history. Hence $Y_n$ is a Markov chain and we can compute the long-term probabilities $\pi_j$ from the usual linear system of Markov chain steady-state equations.

First we calculate the expected probability $q_k$ that $k$ people arrive during the time it takes to service a customer. We split on the possible service times $t$.

$$q_k = \int_0^\infty P[N_t = k]P[\text{service time} = t]dt \quad \text{(imprecise)}$$

$$= \int_0^\infty e^{-\lambda t} \frac{\lambda^t}{k!} g(t)dt.$$ In the discrete case with possible service times $t_1, t_2, \ldots, t_n$ with probabilities $p_i(t)$ we have

$$q_k = \sum_{i=t_1,t_2,\ldots,t_n} e^{-\lambda t} \frac{\lambda^t}{k!} p(t).$$ Now the transition probabilities $T_{ij}$ from a state $Y_n = i$ to a next state $Y_{n+1} = j$.

**Case** $T_{0,0}$, $Y_0 = 0$ and $Y_{n+1} = k$. Since $Y_0 = 0$, there is an expected $1/\lambda$ wait until customer $n+1$ arrives. When he departs, the $k$ people left in the queue must have arrived when he was being served. This has probability $T_{0,k} = q_k$.

**Case** $T_{1,k}$, $Y_1 = 1$ and $Y_{n+1} = k$. In this case customer $n+1$ is the sole member of the $Y_n$ queue and when he departs, the $k$ people left in the queue must have arrived when he was being served. This has probability $T_{1,k} = q_k$.

The transition matrix $T_{ij}$ of probabilities that the queue length between departures goes from $j$ to $k$ is:

| $i$ | $j=0$ | $1$ | $2$ | $3$ | ...
|-----|------|----|----|----|----|
| $0$ | $q_0$ | $q_1$ | $q_2$ | $q_3$ | ...
| $1$ | $q_0$ | $q_1$ | $q_2$ | $q_3$ | ...
| $2$ | $0$   | $q_0$ | $q_1$ | $q_2$ | ...
| $3$ | $0$   | $0$   | $q_0$ | $q_1$ | ...
| ... | ...   | ...   | ...   | ...   | ...

Recall that the long-term probabilities, if they exist, are steady-state probabilities, i.e., they form a probability distribution which remains unchanged from one step to the next. Hence, as before,

$$\pi_0 = \pi_0 q_0 + \pi_1 q_0$$
$$\pi_1 = \pi_0 q_1 + \pi_1 q_1 + \pi_2 q_0$$
$$\pi_2 = \pi_0 q_2 + \pi_1 q_2 + \pi_2 q_1 + \pi_3 q_0$$

Solving and using the Lemma that $\pi_0 = 1-\rho$ gives

$$\pi_0 = (1-\rho)$$
$$\pi_1 q_0 = \pi_0 - \pi_0 q_0 \quad \Rightarrow \quad \pi_1 = (1/q_0)(\pi_0 - \pi_0 q_0)$$
$$\pi_2 = (1/q_0)(\pi_1 - \pi_0 q_1 - \pi_1 q_1)$$
$$\pi_3 = (1/q_0)(\pi_2 - \pi_0 q_2 - \pi_1 q_2 - \pi_2 q_1)$$
$$\pi_n+1 = (1/q_0)(\pi_n - \pi_0 q_n - \pi_1 q_n - \pi_2 q_n - \pi_3 q_n)$$

**Customers arrive at a one-counter bank at a rate of 5 an hour. 90% want to cash a check which takes 5 minutes (1/12 an hour). 10% want to set up a new account which takes 30 minutes (1/2 an hour).** Find the long-term probabilities $\pi_0$, $\pi_1$, $\pi_2$ of having 0, 1, or 2 people standing in line (waiting or being served). $\lambda = 5$, $1/\mu = \text{expected service time} = 5/(.9)+30/(.1)=30/4$. $\mu = \text{service rate} = 4/30 \text{ per minute} = 8 \text{ per hour}$. $\rho = \lambda/\mu = 5/8$. $\mathbb{E} = \sum_{t=0}^{\infty} e^{-0.5 t} / t! - p(t) = e^{-0.5}(.9) + e^{-0.5}(.1) = .6015$ $q_1 = \sum_{t=1}^{\infty} e^{-0.5 t} / t! - p(t)$
\[ e^{-5/12}(5/12)(.9) + e^{-5/2}(5/2)(.1) = .2677 \]
\[ \pi_0 = 1 - \rho = 1 - 5/8 = 3/8 = 37.5\% \]
\[ \pi_1 = (1/q_0)(\pi_0 - \rho q_0) = 24.84\% \]
\[ \pi_2 = (1/q_0)(\pi_1 - \pi_0 q_1 - \pi_1 q_1) = 13.55\% \]

G(x) / Poisson / 1 Queues

Suppose the times between arrivals are independent with cumulative distribution \( G(x) \) and density function \( g(x) \) and rate \( \lambda \) and average time \( 1/\lambda \). Suppose that while there are customers to be served, the service times are a Poisson process with rate \( \mu \) and average time \( 1/\mu \) between arrivals. Suppose there is one server.

Let \( Y_n \) be the number of people in the queue just before the \( n \)th arrival. If there is no one in the queue when the \( n \)th customer arrives, \( Y_n = 0 \). Since the arrivals are independent and the service times are exponential, the queue is memoryless. Hence it is a Markov chain and we can compute the long-term probability \( \pi_i = \lim_{n \to \infty} P[Y_n = i] \) that the queue length will be \( i \) from the usual linear system of Markov chain equations.

As before, let \( \rho = \lambda/\mu \). If \( \lambda > \mu \), then \( \rho > 1 \) and the queue will grow forever. Hence we will assume \( \rho < 1 \). Under rather general regularity conditions, the limit probabilities \( \pi_i \) exist.

Let \( k \) be the number of customers served between the arrival times of \( Y_n \) and \( Y_{n+1} \). Then \( Y_{n+1} = Y_n + 1 - k \) is the \( Y_n \) customers in the queue at the time of the \( n \)th arrival + the \( 1 \) \( n \)th arrival – the \( k \) customers who have been served.

For \( Y_{n+1} > 0 \), let \( q_k \) be the probability that exactly \( k \) customers are served between successive arrivals \( Y_n \) and \( Y_{n+1} \). We split on the possible times \( s \) between arrivals.

\[ q_k = \int_0^\infty P(k \text{ customers are served}|\text{time}=s)P(\text{time between arrivals}=s)ds \]
\[ = \int_0^\infty \frac{e^{-\mu s} (\mu s)^k}{k!} g(s)ds \]

In the case of a discrete set \( \{s_0, s_1, \ldots\} \) of service times with probabilities \( p(s_0), p(s_1), \ldots \) this becomes

\[ \sum_{s \in \{s_0, s_1, \ldots\}} \frac{e^{-\mu s} (\mu s)^k}{k!} p(s) \]

Now calculate the transition probabilities \( T_{ij} \) from a state \( Y_n = i \) to a state \( Y_{n+1} = j \).

Since there are \( Y_n \) customers before the \( n \)th arrival, there are \( Y_n + 1 \) customers just after the \( n \)th arrival. Thus \( Y_{n+1} \leq Y_n + 1 \), in fact \( Y_{n+1} = Y_n + 1 - k \) where \( k \) is the number who get served between the two arrivals.

Case \( j > i + 1 \). \( T_{i,j} = 0 \). This case is impossible since \( j = Y_{n+1} \leq Y_n + 1 = i + 1 \).

Case \( j = i + 1 \). \( T_{i,i+1} = q_0 \). Thus \( Y_n = i, Y_{n+1} = i + 1 \). Customer \( n \) arrives, the queue increases to \( i + 1 \), 0 people get served, the queue still has size \( i + 1 \) when customer \( n + 1 \) arrives. This has probability \( q_0 \).

Case \( 0 < j = i \). \( T_{i,i} = T_{i,i+1} = q_1 \). Thus \( Y_n = i, Y_{n+1} = i \). Customer \( n \) arrives, the queue increases to \( i + 1 \), exactly 1 person gets served, the queue again has size \( i \) when customer \( n + 1 \) arrives. This has probability \( q_1 \).

Case \( 0 < j = i + 1 - k \). \( T_{i,i+1-k} = q_k \).

Customer \( n \) arrives, the queue increases to \( i + 1 \), exactly \( k \) people get served, the queue size is down to \( i + 1 - k \) when customer \( n + 1 \) arrives. This has probability \( q_k \).

Case \( j = 0 \). \( T_{i,0} = r_{i+1} = 1 - (q_0 + q_1 + \ldots + q_i) \).

Thus \( Y_n = i \) and \( Y_{n+1} = 0 \). Customer \( n \) arrives, the queue increases to \( i + 1 \). If the queue size goes down to 0, then all \( i + 1 \) get served. This has some probability \( r_{i+1} \). This isn’t \( q_{i+1} \) since \( q_{i+1} \) was calculated under the assumption that \( Y_{n+1} > 0 \). But if the queue size is \( i + 1 \), then the number \( k \) which must be one of \( 0, 1, \ldots, i + 1 \). Thus

\[ \sum_{j=0}^{i+1} \pi_j q_j = 1 - (q_0 + q_1 + \ldots + q_i) \]

Thus the transition matrix \( T_{jk} \) of probabilities that the queue length between departures goes from \( j \) to \( k \)

<table>
<thead>
<tr>
<th>j</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</tbody>
</table>

\[ (\pi_0, \pi_1, \pi_2, \ldots) = (\pi_0, \pi_1, \pi_2, \ldots) \]

\[ \pi_0 = \pi_0 r_1 + \pi_0 r_2 + \pi_2 r_2 + \pi_3 r_3 + \ldots \]
\[ \pi_1 = \pi_0 q_0 + \pi_0 q_1 + \pi_1 q_1 + \pi_3 q_2 + \pi_2 q_2 + \pi_3 q_3 + \ldots \]
\[ \pi_2 = \pi_1 q_0 + \pi_1 q_1 + \pi_3 q_2 + \pi_3 q_3 + \pi_4 q_2 + \pi_3 q_3 + \ldots \]
\[ \pi_3 = \pi_2 q_0 + \pi_3 q_1 + \pi_4 q_2 + \pi_3 q_3 + \ldots \]

**Lemma.** If \( \rho = \lambda/\mu < 1 \), \( \pi_j = (1-\beta)\beta^j \) where \( \beta \in (0,1) \) is the solution of the equation:

\[ \beta = q_0 + q_1 \beta + q_2 \beta^2 + q_3 \beta^3 + \ldots \]

Since \( \pi_0 + \pi_1 + \pi_2 + \ldots = 1 \), \( \pi_0 = 1 - (\pi_1 + \pi_2 + \ldots) \).

Since \( r_{i+1} = 1 - (q_0 + q_1 + \ldots + q_i) \), \( r_1 = 1 - q_0 \), \( r_2 = 1 - (q_0 + q_1) \), \( r_3 = 1 - (q_0 + q_1 + q_2) \). \ldots

Hence the first line \( \pi_0 = \pi_0 r_1 + \pi_1 r_2 + \pi_2 r_3 + \ldots \) is just 1 minus the sum of the remaining lines and hence is dependent.

Thus we need only show that \( \pi_j = (1-\beta)\beta^j \) is a solution for the remaining equations. The third line is typical.

To prove

\[ \pi_2 = \pi_1 q_0 + \pi_2 q_1 + \pi_3 q_2 + \ldots \]
Start with the right side and use definition of $\beta$ and $\pi_i$.

$$
\pi_1 q_0 + \pi_2 q_1 + \pi_3 q_2 + ... \\
= (1 - \beta) \beta^0 q_0 + (1 - \beta) \beta^1 q_1 + (1 - \beta) \beta^2 q_2 + ... \\
= (1 - \beta) [q_0 + \beta^0 q_1 + \beta^1 q_2 + ...] \\
= (1 - \beta) [q_0 + q_1 \beta + q_2 \beta^2 + ...] \\
= (1 - \beta) [\beta^0] = (1 - \beta) \beta^2 = \pi_2
$$