Name: _______________________________  Student ID: __________________________

**Instructions.** Write your name and your student id number in the spaces provided above. Each problem is worth 12 points. Write your answers in the spaces provided on this exam. Do not use your own paper. If you need scratch paper, use the back pages of the exam. You must justify your answers to receive full credit. You may use a calculator. This exam is closed book and closed notes.

1. Complete the following sentences so that each one becomes an informal definition of the word in italics:
   (a) A *proposition* is a declarative sentence that is either true or false, but not both.
   (b) A *predicate* is a declarative sentence that describes properties of its subject or subjects.

2. State *The Fundamental Theorem of Arithmetic*.

   Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

3. Write each of the following statements in the form “if . . . , then . . . ”:
   (a) $x$ is even only if $y$ is odd.
      If $x$ is even then $y$ is odd.
   (b) $A$ implies $B$.
      If $A$ then $B$.
   (c) It is hot whenever it is sunny.
      If it is sunny then it is hot.
   (d) To get a good grade it is necessary that you study.
      If you get a good grade then you study.
   (e) Studying is sufficient for passing.
      If you study then you pass.
   (f) You need to be registered in order to check out library books.
      If you can check out library books then you are registered.
4. Let $F(A)$ be the predicate “$A$ is a finite set”, $I(A)$ the predicate “$A$ is an infinite set”, $S(A,B)$ the predicate “$A$ is a subset of $B$”, and $E(A)$ the predicate “$A$ is empty”. Suppose the universe of discourse consists of all subsets of $\mathbb{R}$. Using these predicates together with logical operators and quantifiers, translate the following propositions into logical expressions:

(a) Not all sets are finite.
$$\neg \forall A \: F(A).$$

(b) Every subset of a finite set is finite.
$$\forall A \forall B \left[ F(B) \land S(A,B) \right] \rightarrow F(A).$$

(c) No infinite set can be a subset of a finite set.
$$\neg (\exists A \exists B \: I(A) \land F(B) \land S(A,B)) \equiv \forall A \forall B \left[ I(A) \land F(B) \right] \rightarrow \neg S(A,B)$$

(d) The empty set is a subset of every finite set.
$$\forall B \: S(\emptyset,B).$$

5. (a) Show that 97 is a prime number.

Note that $9 < \sqrt{97} < 10$, so it suffices to check whether the primes less than or equal to 9 divide 97. Note that $2 \nmid 97$, $3 \nmid 97$, $5 \nmid 97$, and $7 \nmid 97$: therefore 97 is a prime number.

(b) Find the prime factorization of 249998294.
$$249998294 = 2 \cdot 7^3 \cdot 13 \cdot 17^2 \cdot 97.$$ 

(c) Use the Euclidean algorithm to find the gcd(450, 120).
$$450 = 3 \cdot 120 + 90, \text{ so } \gcd(450,120) = \gcd(120,90). \quad 120 = 1 \cdot 90 + 30, \text{ so } \gcd(120,90) = \gcd(90,30) = 30. \text{ Therefore, } \gcd(450,120) = 30.$$

6. Observe that $69 \cdot 164 - 31 \cdot 365 = 1$.

(a) Solve $164 \cdot x \equiv 1 \pmod{365}$, $0 \leq x \leq 364$.
$$69 \cdot 164 \equiv 1 \pmod{365}, \text{ so } x = 69.$$

(b) Solve $31 \cdot x \equiv 1 \pmod{164}$, $0 \leq x \leq 163$.
$$1 \equiv 31 \cdot (-365) \equiv 31 \cdot (3 \cdot 164 - 365) \equiv 31 \cdot 127 \pmod{164}, \text{ so } x = 127.$$

(c) Solve $9 \cdot x \equiv 1 \pmod{31}$, $0 \leq x \leq 30$.
$$1 \equiv 69 \cdot 164 \equiv (69 - 62) \cdot (164 - 155) \equiv 7 \cdot 9 \pmod{31}, \text{ so } x = 7.$$

7. Suppose $B = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 \\ 0 & 6 \end{bmatrix}$. Find a matrix $A$ such that $AB = C$. (Hint: right multiply both sides of this equation by $B^{-1}$.)

$$A = CB^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{3 \cdot 4 - 2 \cdot 5} \begin{bmatrix} 2 & 1 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 6 & -7 \\ -12 & 18 \end{bmatrix} = \begin{bmatrix} 3 & -7/2 \\ 6 & 9 \end{bmatrix}.$$
8. What is the rule of inference used in each of the following:

(a) If it snows today, the university will be closed. The university will not be closed today. Therefore, it did not snow today.

   Modus tollens.

(b) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.

   Hypothetical syllogism.

9. Determine whether the following argument is valid. Justify your answer.

   \[ p \rightarrow r \]
   \[ q \rightarrow r \]
   \[ \neg(p \lor q) \]
   \[ :\neg r \]

   The argument is invalid since the values \( p = F, q = F, r = T \) make all of the premises true and the conclusion false.

10. Find the best big-oh notation to describe the number of print statements in the following code. Choose your answer from the following: \( 1 \), \( \log n \), \( n \), \( n \log n \), \( n^2 \), \( n^3 \), \( \ldots \), \( 2^n \), \( n! \).

    ```
    i := 1;
    while i <= n
        begin
            j:= 1;
            while j <= i
                begin
                    print "hello";
                    j := j + 1
                end
            i := i + 1
        end
    ```

    The print statement is executed \( 1 + 2 + \cdots + n = n(n+1)/2 \) times, which belongs to the class \( O(n^2) \).

11. Give a proof by contradiction of the following: “If \( n \) is an odd integer, then \( n^2 \) is odd.”

    Let \( n \) be an odd integer and assume that \( n^2 \) is an even integer. Then \( 2 \mid (n + 1) \) and \( 2 \mid n^2 \).
    Since \( n = n(n + 1) - n^2 \), we conclude that \( 2 \mid n \) and therefore \( n \) is an even integer. This contradicts the hypothesis that \( n \) is an odd integer, so the assumption that \( n^2 \) is even must be false. Thus we have proved that when \( n \) is an odd integer then \( n^2 \) is an odd integer.

12. Suppose that a “word” is any string of 7 letters of the alphabet, with repeated letters allowed.

   (a) How many words are there?

      There are \( 26^7 \) words.

   (b) How many words with no repeated letters are there?

      There are \( P(26, 7) \) words with no repeated letters.
(c) How many words that begin with “R” and end with “T” are there?
There are $26^5$ words that begin with “R” and end with “T”.

(d) How many words that begin with “R” or end with “T” are there?
There are $26^6 + 26^6 - 26^5 = (2 \cdot 26 - 1)26^5 = 51 \cdot 26^5$ words that begin with “R” or end with “T”.

(e) How many words begin with “A” or “B”?
There are $26^6 + 26^6 = 2 \cdot 26^6$ words that begin with “A” or “B”.

(f) How many words have exactly one vowel?
There are $26^7 - 21^7$ words that have exactly one vowel.

13. Let $x$, $y$ and $z$ be nonnegative integers.
(a) Find the number of solutions to $x + y + z = 32$.
There are $C(34, 32) = C(34, 2)$ solutions.

(b) Answer part (a), but assume $x \geq 7$ and $y \geq 15$.
There are $C(12, 10) = C(12, 2)$ solutions.

(c) Answer part (a), but assume $z \leq 2$.
There are $C(34, 32) - C(31, 29) = C(34, 2) - C(31, 2)$ solutions.

14. A factory makes automobile parts. Each part has a code consisting of a digit, a letter, and a digit with the digits distinct, such as 5C7, 1O6, 3Z0. Last week the factory made 5000 parts. Find the minimum number of parts that must have the same serial number.
There are $10 \cdot 26 \cdot 9 = 2340$ distinct codes. Since $\lceil 5000/2340 \rceil = 3$, at least one code must appear on at least 3 manufactured parts.

15. Let $P$ be a probability function for the sample space $S$.
(a) Let $A_1 \subseteq S$ and $A_2 \subseteq S$ be two events, and let $A_1A_2 = A_1 \cap A_2$ denote the intersection of these events. State the multiplication rule for the probability of the intersection.

$$P(A_1A_2) = P(A_1) \cdot P(A_2 \mid A_1).$$

(b) Let $n \geq 2$ be an integer and define the predicate $Q(n)$ as follows:

$$Q(n) : P(A_1A_2 \cdots A_{n-1}A_n) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1A_2)P(A_4 \mid A_1A_2A_3) \cdots P(A_n \mid A_1A_2A_3 \cdots A_{n-1})$$

Use mathematical induction to prove the generalized multiplication rule; in other words prove that $Q(n)$ is true for all $n \geq 2$. (Hint: the Basis Step is covered in part (a).)

Basis Step. When $n = 2$, the proposition $Q(2)$ says $P(A_1A_2) = P(A_1) \cdot P(A_2 \mid A_1)$, which is the multiplication rule from part (a). Therefore, $Q(n)$ is true for $n = 2$. 
17. An urn has three green balls and five red balls.  

(a) Two balls are drawn from the urn without replacement. What is the probability that the first ball is red and the second ball is green?  

\[
P(\text{First Red and Second Green}) = P(\text{First Red})P(\text{Second Green} \mid \text{First Red}) = (5/8)(3/7) = 15/56.
\]

(b) What is the probability that the second ball is green?  

\[
\]

18. Let \( S \) be a finite sample space and let \( P \) be a probability function for \( S \).  

(a) State the three Kolmogorov axioms for the probability function \( P \).  

\[  
\textbf{A1: } P(S) = 1.  
\textbf{A2: } \text{For every event } A \subseteq S, P(A) \geq 0.  
\textbf{A3: } \text{For every pair of events } A_1, A_2 \subseteq S, \text{ if } A_1 \cap A_2 = \emptyset \text{ then } P(A_1 \cup A_2) = P(A_1) + P(A_2).  
\]
(b) Fix an event \( B \subseteq S \) such that \( P(B) \neq 0 \). Let \( A \subseteq S \) be any event. Define the conditional probability

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)}.
\]

(c) Define a new function, \( \mathcal{P} \), on events \( A \subseteq S \) by \( \mathcal{P}(A) = P(A \mid B) \). Show that \( \mathcal{P} \) satisfies the Kolmogorov axioms.

\[\textbf{A1:} \quad \text{Since } B \subseteq S, \text{ we know that } S \cap B = B. \text{ Hence } \mathcal{P}(S) = P(S \mid B) = P(S \cap B)/P(B) = P(B)/P(B) = 1; \mathcal{P} \text{ satisfies the first Kolmogorov axiom.}\]

\[\textbf{A2:} \quad \text{Given any event } A \subseteq S, \text{ we know that } P(A \cap B) \geq 0. \text{ Since } P(B) > 0, \mathcal{P}(A) = P(A \cap B)/P(B) \geq 0; \mathcal{P} \text{ satisfies the second Kolmogorov axiom.}\]

\[\textbf{A3:} \quad \text{Let } A_1, A_2 \text{ be any pair of mutually exclusive events in } S. \text{ Since } A_1 \cap A_2 = \emptyset, (A_1 \cap B) \cap (A_2 \cap B) = (A_1 \cap A_2) \cap B = \emptyset \cap B = \emptyset, \text{ we see that } A_1 \cap B \text{ and } A_2 \cap B \text{ are mutually exclusive events. Thus}\]

\[
\mathcal{P}(A_1 \cup A_2) = P((A_1 \cup A_2) \cap B)/P(B) = P((A_1 \cap B) \cup (A_2 \cap B))/P(B)
\]

\[
= (P(A_1 \cap B) + P(A_2 \cap B))/P(B) = P(A_1 \cap B)/P(B) + P(A_2 \cap B)/P(B)
\]

\[
= \mathcal{P}(A_1) + \mathcal{P}(A_2).
\]

\( \mathcal{P} \) satisfies the third Kolmogorov axiom.

19. An assembler of electric fans uses motors from two sources. Company \( A \) supplies 90 percent of the motors and company \( B \) supplies the other 10 percent of the motors. Suppose it is known that 5 percent of the motors supplied by company \( A \) are defective and 3 percent of the motors supplied by company \( B \) are defective. An assembled fan is found to have a defective motor. What is the probability that this motor was supplied by company \( B \)?

Let \( A \) be the event that a motor is manufactured by company \( A \), let \( B \) be the event that a motor is manufactured by company \( B \), and let \( D \) be the event that a motor is defective. The following probabilities are given:

\[
P(A) = 0.9, \quad P(B) = 0.1, \quad P(D \mid A) = 0.05, \quad P(D \mid B) = 0.03.
\]

From this we can compute the probability that a motor is defective,

\[
P(D) = P(A)P(D \mid A) + P(B)P(D \mid B) = 0.045 + 0.003 = 0.048,
\]

and we can compute the probability that a motor is defective and comes from company \( B \);

\[
P(B \cap D) = P(B)P(D \mid B) = 0.003.
\]

Using these computed probabilities, we find the probability that a defective motor is supplied by company \( B \) to be:

\[
P(B \mid D) = P(B \cap D)/P(D) = 0.003/0.048 = 3/48 = 1/16 = 0.0625.
\]
20. A fair die is rolled one time. Consider the events:
   • A: observe an odd number
   • B: observe an even number
   • C: observe a 1 or a 2
   (a) Are A and B independent events?
   
   A and B are mutually exclusive events, so \( P(A \cap B) = 0 \). Since \( P(A) = 1/2 \) and \( P(B) = 1/2 \), \( P(A \cap B) \neq P(A)P(B) \), so A and B are not independent events.

   (b) Are A and C independent events?
   
   \( P(A) = 1/2 \), \( P(C) = 1/3 \), and \( P(A \cap C) = 1/6 = P(A)P(B) \), so A and B are independent events.