Why Probability?

- In the real world, we often don't know whether a given proposition is true or false.
- Probability theory gives us a way to reason about propositions whose truth is uncertain.
- Useful in weighing evidence, diagnosing problems, and analyzing situations whose exact details are unknown.

Also some ability with probability theory is required in ICS 311 (Algorithms).

The Rosen text treats probability in 3 sections of Chapter 5.

Here, in these notes, probability is treated in 4 lectures, and not necessarily in the same order as is treated in the text.

Our 4 lectures are:

1. Introduction
2. Conditional probability
3. Random variables and their properties
4. Some special random variables, the Bernoulli, Binomial, Geometric, and Poisson distributions
PROBABILITY

- Probability is a real-valued set function
- "Notation is everything"
- The Basic Concepts
  Experiment, Sample Space, Event
- SAMPLE SPACE of Possible Outcomes of a Chance Experiment, \( S = \{\omega_1, \omega_2, \ldots, \omega_K\} \)
- Examples
  \( E_a : \) Coin toss, \( S = \{\omega_1, \omega_2\} = \{H, T\}, \ |S| = 2 \)

  \( E_b : \) 3 Successive Coin Tosses,
  \( S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\} \)
  \( S = \{\omega_1, \omega_2, \ldots, \omega_8\}, \ |S| = 8 \)

  \( E_c : \) Dice Toss. \( S = \{\text{faces in dice toss}\} \)
  \( S = \{1, 2, 3, 4, 5, 6\}, \ |S| = 6 \)
PROBABILITY: EVENTS

EVENT: A set of outcomes on the sample space S

- \( E_b : 3 \text{ Successive Coin Tosses} \)

\[ S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\} \]
\[ S = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\} \]

\[ A = \{1\text{st coin is } H\} = \{\omega_1, \omega_2, \omega_3, \omega_7\}, \quad |A| = 4 \]

\[ B = \{\text{exactly 2H's}\} = \{\omega_2, \omega_3, \omega_4\}, \quad |B| = 3 \]

\[ C = \{\text{same face on the 1st and 3rd toss}\} = \{\omega_2, \omega_3, \omega_6, \omega_8\} \]
\[ = \{HHH, HTH, THT, TTT\}, \quad |C| = 4 \]

- \( E_d : 2 \text{ Dice Tosses} \)

\[ S = \{(1, 1), (1, 2), \ldots, (1, 6), (2, 1), \ldots, (2, 6), \ldots, (6, 6)\} \]

\[ A = \{\text{the sum of 2 faces is 2}\} = \{(1, 1)\}, \quad |A| = 1 \]

\[ B = \{\text{the sum of 2 faces is 3}\} = \{(1, 2), (2, 1)\}, \quad |B| = 2 \]

\[ C = \{\text{the sum of 2 faces is 4}\} = \{(1, 3), (2, 2), (3, 1)\} \]
UNIFORM DISCRETE PROBABILITY SPACE or, EQUALLY LIKELY OUTCOMES

DEFINITION: The probability of an event \( E \) in a space \( S \) of equally likely outcomes is

\[
P(E) = \frac{|E|}{|S|}
\]

• An urn contains 4 blue and 5 red balls. What is the probability that a ball chosen from the urn is blue?

\[
S = \{b_1, b_2, b_3, b_4, r_1, r_2, r_3, r_4, r_5\}, \quad |S| = 9
\]

\[
A = \{\text{a blue ball is chosen}\} = \{b_1, b_2, b_3, b_4\}
\]

\[
|A| = 4, \quad |S| = 9, \quad P(A) = \frac{|A|}{|S|} = \frac{4}{9}
\]
UNIFORM PROBABILITY SPACE, EXAMPLES

- $E_d$ : 2 Dice Tosses,
  \[ S = \{\omega_1, \omega_2, \ldots, \omega_{36}\}, \quad |S| = 36 \]
  
  **ASSUME**
  
  \[ \text{Probability}(\omega_i) = P(\omega_i) = \frac{1}{36} \quad i = 1, 2, \ldots, 36 \]
  
- $A = \{\text{the sum of two faces} = 2\} = \{(1,1)\}$
  \[ P(A) = \frac{\# \text{ ways favorable to } A}{\# \text{ possible (equally likely ) outcomes}} = \frac{|A|}{|S|} = \frac{1}{36} \]

- $B = \{\text{the sum of two faces} = 3\} = \{(1,2),(2,1)\}$
  \[ P(B) = \frac{\# \text{ ways favorable to } B}{\# \text{ possible (equally likely ) outcomes}} = \frac{|B|}{|S|} = \frac{2}{36} \]

- $A_k = \{\text{the sum of two faces} = k\}, \ k = 2, 3, \ldots, 12$
  \[ P(A_k) = \frac{k - 1}{36}, \quad k = 2, 3, \ldots, 7 \]
Table 1: The Sum of Two Dice

<table>
<thead>
<tr>
<th>First die</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 2: Table of Probabilities of the sum of 2 dice

<table>
<thead>
<tr>
<th>Sum</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(A_k)$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\frac{4}{36}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{6}{36}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{4}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{1}{36}$</td>
</tr>
</tbody>
</table>

Note that $|A_2| = |A_{12}|$, $|A_3| = |A_{11}|$, $|A_4| = |A_{10}|$, $|A_5| = |A_9|$, $|A_6| = |A_8|$
UNIFORM PROBABILITY SPACE, POKER EXAMPLES

• POKER 
  \[ S = \{\omega_1, \omega_2, \ldots, \omega_{C(52, 5)}\}, |S| = C(52, 5) \]

  \[ \text{ASSUME } P(\omega_i) = \frac{1}{C(52, 5)} \]
  \[ C(52, 5) = 2,598,960 \]

• E = Four of a kind = \( E_1 \cap E_2 \)

  \[ P\{4'\text{of a kind}\} = \frac{|4 \text{ of a kind hands}|}{|\text{poker hands}|} \]

  \[ E_1 = \{4 \text{ of a kind varieties}\} \]
  \[ E_1 = \{A, 2, \ldots, 10, J, Q, K\}, \quad |E_1| = 13 = C(13, 1) \]

  \[ E_2 = \{\text{other cards}\}, \quad |\text{other cards}| = (52 - 4) = 48 = C(48, 4) \]

  \[ C(52, 5) = \frac{52!}{47!5!} = 2,598,960 \]

  \[ P\{4 \text{ of a kind}\} = \frac{13 \times 48}{C(52, 5)} \approx 0.00024 \]
What is the probability that a poker hand contains a full house?

(3 of one kind and 2 of another kind)

Let $A = \text{full house}$.

$A_1 = \{\text{choose 2 card types out of 13}\}, |A_1| = P(13, 2) = 156$

$A_2 = \{\text{choose 3 of a kind}\}, \quad |A_2| = C(4, 3) = 4$

$A_3 = \{\text{choose 2 of a kind}\}, \quad |A_3| = C(4, 2) = 6$

$A = A_1 \cap A_2 \cap A_3$

$P(A) = \frac{|A|}{|S|}$

$P(A) = \frac{|A_1| \times |A_2| \times |A_3|}{|S|} = \frac{156 \times 4 \times 6}{C(52, 5)} \approx 0.0014$
Exercise 5.1.15  P361

What is the probability that a five-card poker hand contains two pairs?

E = # 2 pairs are chosen in a poker hand

The number of ways to hold 2 pairs

\[
C\left(\frac{13}{2}\right) = \binom{13}{2} = 78 \text{ ways of choosing 2 kinds of pairs.}
\]

\[
\binom{4}{2} = 6 \text{ ways of choosing 2 suits from 4 suits}
\]

44 remaining cards to choose from

4 chosen, do not choose one from the pairs (exclude 8 total)

Then using \( P(E) = \frac{|E|}{151} \)

\[
\# \text{ pairs} / 2 \text{ suits from pair 1} / 2 \text{ suits from pair 2}
\]

\[
P(E) = \frac{78 \times 6 \times 6 \times 44}{\binom{\frac{52}{2}}} = \frac{123552}{2,598,960} = 0.04758902
\]
A Uniform Probability Space Example

**Problem:** Roll 3 dice, let \( E = \{ \text{exactly 2 faces } \geq 5 \} \), Compute \( P(E) \).

**Solution:** Let \( A = \{ \text{outcomes of 1 die toss} \} = \{\omega_1, \omega_2, \ldots, \omega_6\} \)

Let \( S = \frac{4}{6}A \times A \times A \) (The Cartesian product)

\[
S = \{\omega_1, \omega_1, \ldots, \omega_6 \times 6 \times 6\}, \quad |S| = 216 = 6^3
\]

\( P(\omega_i) = 1/216, \quad i = 1, 2, \ldots, 216 \)\)

Let \( E = \{ \text{exactly 2 faces } \geq 5 \} = \{55x, 56x, 65x\} \)

\[
E = E_1 \cup E_2 \cup E_3
\]

\[
|E| = |E_1| + |E_2| + |E_3|
\]

\( E_1 = \{ \text{3 choices for placement of x, 4 choices of x that are not 5 or 6} \} \)

\[
|E_1| = 3 \times 4, \quad |E_2| = 3 \times 4, \quad |E_2| = 3 \times 4 \times 2
\]

\[
|E| = 3 \times 4 \times 4 = 48
\]

\[
P\{\text{exactly 2 faces } \geq 5 \text{ in a dice roll}\} = \frac{\text{number of times exactly 2 faces } \geq 5}{\text{number of outcomes in 3 tosses}}
\]

\[
P(E) = \frac{|E|}{|S|} = \frac{48}{216} = 0.22222
\]
PROBABILITY IS A REAL VALUED SET FUNCTION
NOTATION IS EVERYTHING

AXIOMS OF PROBABILITY

1. \( P(S) = 1 \) \( \quad (\frac{|S|}{|S|} = 1) \)

2. \( A \subset S, \quad P(A) \geq 0 \) \( \quad (\frac{|A|}{|S|} \geq 0) \)

3. \( A \subset S, \quad B \subset S, \quad A \cap B = \emptyset, \quad P(A \cup B) = P(A) + P(B) \)

\[ E(A \cup B) \neq \emptyset \quad |E| = |A| + |B| \]

\[ P(A \cup B) = P(E) = |E| = \frac{|A| + |B|}{|S|} = \frac{|A|}{|S|} + \frac{|B|}{|S|} = P(A) + P(B) \]

3'. \( A_i \subset S, \quad i = 1, 2, \ldots, n, \quad A_i \cap A_j = \emptyset, \quad i \neq j \)

\[ P(\bigcup_{i=1}^{n} A_i) = P(A_1) + P(A_2) + \ldots + P(A_n) \]
THE PROBABILITY OF COMBINATION OF EVENTS

**Theorem 1** P 359

- **Theorem:** \( S = \bar{A} \cup \bar{A} \), \( P(\bar{A}) = 1 - P(A) \)
- **Proof:** (Use Axiom 3)

\[ S = A \cup \bar{A}, \quad A \cap \bar{A} = \emptyset \quad \text{then} \]
\[ P(S) = P(A) + P(\bar{A}) \quad \text{(Axiom 3)} \]

\[ P(\bar{A}) = P(S) - P(A) \]

\[ P(\bar{A}) = 1 - P(A) \quad \text{(Axiom 1)} \]

**Corollary**

\[ P(\emptyset) = 0 \]
\[ P(\emptyset) = 1 - P(\bar{\emptyset}) \]
\[ = 1 - P(S) \]
\[ = 1 - 1 = 0 \]
EXAMPLE 9

A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

\[ S = \{ 10 \text{ equally likely 10 bit words} \}, \quad |S| = 2^{10} = 1024 \]

\[ E = \{ \text{at least one bit out of 10 is '0'} \} \]

\[ \bar{E} = \{ \text{all 10 bits are '1'} \} \]

\[ \mathcal{S} = E \cup \bar{E} \]

\[ \mathcal{E} \cap \bar{E} = \emptyset \]

\[ P(E) = 1 - P(\bar{E}) \]

\[ P(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{1}{1024} \]

\[ P(E) = 1 - \frac{1}{1024} = \frac{1023}{1024} \approx 0.9990 \]
\textbf{THEOREM}: \( A \subset S, \quad B \subset S \quad (P359) \)

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

\textbf{Remark}: Similar to inclusion-exclusion rule

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

Proof: (Use Axiom 3)

\[ A \cup B = AB \cup AB \cup \bar{A}B \]

\[ P(A \cup B) = P(AB) + P(AB) + P(\bar{A}B) \quad \text{Axio\textsuperscript{m} 3} \]

\[ A = AB \cup AB, \quad P(A) = P(AB) + P(AB) \quad \Rightarrow \quad P(AB) = P(AB) - P(AB) \quad \text{Axio\textsuperscript{m} 3} \]

\[ B = \bar{A}B \cup \bar{A}B, \quad P(B) = P(\bar{A}B) + P(AB) \quad \Rightarrow \quad P(AB) = P(AB) - P(AB) \quad \text{Axio\textsuperscript{m} 3} \]

\[ P(A \cup B) = P(A) - P(AB) + P(AB) + P(B) - P(AB) \]

\[ P(A \cup B) = P(A) + P(B) - P(AB) \]

\textbf{Notation}: \( AB = A \cap B \)
What is the probability that a bridge hand (13 cards) contains 4 aces or 4 kings.

The total number of bridge hands defines the sample space:

$$|S| = \binom{52}{13}$$

Counting the number of ways of getting 4 aces: $$\binom{48}{9}$$
Four cards are "forced" (the aces), 9 cards will be chosen (at random from the remaining 48 cards. Similarly for the number of ways of getting 4 kings. Also, the number of ways of getting 4 aces and 4 kings is $$\binom{44}{5}$$

Now, let $E = \{4 \text{ aces or 4 kings}\}$, $E_1 = \{4 \text{ aces}\}$, $E_2 = \{4 \text{ kings}\}$, $E_1 \cap E_2 = \{4 \text{ aces and 4 kings}\}$

$$P(E) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$P(E) = \frac{\binom{48}{9}}{|S|} + \frac{\binom{48}{9}}{|S|} - \frac{\binom{44}{5}}{|S|}$$
EXAMPLE 10

What is the probability that the event $E = \{\text{a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5}\}$?

$S = \{1, 2, \ldots, 100\} = \{n|n \in \mathbb{Z}^+, 1 \leq n \leq 100\}$, $|S| = 100$

$|E_1| = \lfloor \frac{100}{2} \rfloor$

$E_1 = \{2|n\}$, $E_2 = \{5|n\}$

$E_1 \cup E_2 = \{\{2|n\} \cup \{5|n\}\}$,

$E_1 \cap E_2 = \{\{2|n\} \cap \{5|n\}\}$

$P(E) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$P(E) = \frac{|\{2|n\}|}{|S|} + \frac{|\{5|n\}|}{|S|} - \frac{|\{2|n\} \cap \{5|n\}|}{|S|}$

$P(E) = \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{60}{100} = 0.6$
Some Elementary Probability Examples

A box contains 100 light bulbs, 5 of which are defective. If 20 are selected, what is the probability that they all work?

Consider the experiment of flipping four unbiased coins. (On each coin the outcome must be “heads” or “tails”.) Describe a uniform probability space which models this experiment. Compute the probability of “at least two heads”.
Exercises 5.1

1. What is the probability that a card selected from a deck is an ace?

3. What is the probability that a randomly selected integer chosen from the first 100 positive integers is odd?

5. What is the probability that the sum of the numbers on two dice is even when they are rolled?

7. What is the probability that a coin lands heads up six times in a row?
9. What is the probability that a five-card poker hand does not contain the queen of hearts?

11. What is the probability that a five-card poker hand contains the two of diamonds, the three of spades, the six of hearts, the ten of clubs, and the king of hearts?

13. What is the probability that a five-card poker hand contains at least one ace?
15. What is the probability that a five-card poker hand contains two pairs (that is, two of each of two different kinds and a fifth card of a third kind)?

21. What is the probability that a die never comes up an even number when it is rolled six times?
23. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

25. Find the probability of winning the lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding
   - a) 50.   - b) 52.   - c) 56.   - d) 60.

27. Find the probability of selecting exactly one of the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding
   a) 40.   b) 48.   c) 56.   d) 64.
REVIEW: PROBABILITY

PROBABILITY IS A REAL VALUED SET FUNCTION

"Notation is everything"

S is a SAMPLE SPACE of all possible outcomes
(of an experiment)

Subsets of S are called EVENTS, \((A \subseteq S)\)

A probability on S is a function \(P\) that assigns to each event
\(A \subseteq S\) a number \(P(A)\) in \([0, 1]\) that satisfies:

THE AXIOMS OF PROBABILITY

- \(1. \ P(S) = 1\)
- \(2. \ A \subseteq S \rightarrow P(A) \geq 0\)
- \(3. \ A, B \subseteq S, \ A \cap B = \phi \rightarrow P(A \cup B) = P(A) + P(B)\)
- \(3'. \ A_i \subseteq S, \ i = 1, 2, \ldots, n, \ A_i \cap A_j = \phi \rightarrow P(\bigcup_{i=1}^{n} A_i) = P(A_1) + P(A_2) + \ldots + P(A_n)\)

Some consequences of the axioms:

- \(S = A \cup \overline{A} \rightarrow P(\overline{A}) = 1 - P(A)\)
- \(P(\emptyset) = 0\)
- \(P(A \cup B) = P(A) + P(B) - P(A \cap B)\)
CONDITIONAL PROBABILITY

- \( n = |\{\text{population}\}| \)
- \( n_f = |\{\text{females}\}| \)
- \( n_c = |\{\text{colorblind}\}| \)
- \( n_{fc} = |\{\text{colorblind}\text{females}\}| \)

**QUESTION:** What is the probability that a female chosen at random is colorblind?

- **NOTATION:** \( P(C|F) = \frac{n_{fc}}{n_f} = \frac{n_{fc}/n}{n_f/n} = \frac{P(FC)}{P(F)} \)

- **DEFINITION:** \( A, B \subset S \)

\[
P(A|B) = \frac{P(AB)}{P(B)} \quad (P(B) \neq 0)
\]

\[
P(B|A) = \frac{P(AB)}{P(A)} \quad (P(A) \neq 0)
\]

- **MULTIPLICATION RULE**

\[
P(AB) = P(A|B) = P(A)P(B|A) = P(B)P(A|B)
\]
EXAMPLE

- Urn contents 3 Rd(R), 1 Green(G) balls.
- Select 2 balls without replacement.
- Q1: P(both balls are red)?
- Q2: P(second ball selected is red)?

- \( S = \{R_1, R_2, R_3, G\} \)
- Let \( R_i = \{i\text{th ball selected is } R\} \) (slightly inconsistent notation) 
  \[ R_1 = \{1\text{st ball is } R\}, \quad R_2 = \{2\text{nd ball is } R\} \]
- A1: \( P(R_1, R_2) = P(R_1)P(R_2|R_1) = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2} \)
- A2: \{second ball selected is red\}

\[
R_2 = R_1R_2 \cup GR_2 \\
P(R_2) = P(R_1R_2) + P(GR_2) \quad (Axiom3) \\
P(R_2) = P(R_2|R_1)P(R_1) + P(G)P(R_2|G) \\
= \frac{2}{3} \cdot \frac{3}{4} + \frac{1}{4} \cdot 1 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}
\]
**A Conditional Probability Example**

**Problem:** What is the conditional probability that a family with two children has two boys, given that they have at least one boy? Assume that each of the possibilities BB, BG, GB, and GG is equally likely, where B represents a boy and G represents a girl.

\[\begin{array}{c}
B \\
G
\end{array}\]

**Solution:**

\[S = \{\text{BB, BG, GB, GG}\}, \quad |S| = 4\]

\[S = \{\omega_1, \omega_2, \omega_3, \omega_4\}, \quad P(\omega_i) = 1/4, \quad i = 1, \ldots, 4\]

\[A = \{\text{family with 2 children has 2 boys}\} = \{BB\}\]

\[C = \{\text{family has at least 1 boy}\} = \{BB, BG, GB\}\]

\[P(A \mid C) = \frac{P(AC)}{P(C)} = \frac{P(BB)}{P(BB, BG, GB)} = \frac{1/4}{3/4} = \frac{1}{3}\]
EXAMPLE 3  P 3.6c
What is the probability that a bit string of length four, generated at random so that each
of the 16 bit strings of length four is equally likely, contains at least two consecutive
0s, given that its first bit is a 0? (We assume that 0 bits and 1 bits are equally likely.)

\[ S = \text{\{binary bit strings, length 4\}}, \quad |S| = 2^4 = 16 \]
\[ S = \{w_1, w_2, \ldots, w_{16}\}, \quad P(w_i) = \frac{1}{16}, \quad i = 1, \ldots, 16 \]

\[ A = \{\text{at least 2 consecutive 0s in 4 bits}\} \]
\[ B = \{1st \text{ bit is '0'}, \text{ in 4 bit strings}\} \]

\[ B = \{0 U x x x\} = \{00 U 01, \ldots\} \]

\[ |B x 1| = 2^2, \quad P(B) = \frac{|B x 1|}{|S|} = \frac{8}{16} = \frac{1}{2} \]

\[ P(A|B) = \frac{P(AB)}{P(B)} \quad \text{Definition} \]

Enumerate & AB
\[ AB = \{0000, 0001, 0010, 0011, 0100\} \]
\[ |AB| = 5 \]
\[ P(AB) = |AB| / |S| = \frac{5}{16} \]

\[ P(AB) = \frac{5/16}{8/16} = \frac{5}{8} \]
MORE CONDITIONAL PROBABILITY

DEFINITION

\[ P(A|B) = \frac{P(AB)}{P(B)} \]

MULTIPLICATION RULE

\[ P(AB) = P(A|B)P(B) = P(B|A)P(A) = P(A)P(B|A) \]

Then \( P(A_1A_2) = P(A_1)P(A_2|A_1) \)

Also \( P(A_1A_2A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2) \)

Proof: \( P(A_1A_2A_3) = P(A_1A_2)P(A_3|A_1A_2) \)
\[ = P(A_1)P(A_2|A_1)P(A_3|A_1A_2) \]

Generally: \( P(A_1A_2, \ldots A_k) = P(A_1)p(A_2|A_1), \ldots P(A_k|A_1A_2, k-1) \)

EXAMPLE

The first 3 digits of the UH phone exchange is 956.
If the digits are equally likely in the sequence of the remaining 4 digits, what is the probability that a randomly selected UH phone number contains 7 distinct digits?
EXAMPLE 1:
Survey 100 computer installations
75 have brand X computer
3 installations are inspected at random
Question: What is the probability that all installations have brand X?
A Sequence of Conditional Probability Problems

Consider a special deck of 4 cards consisting of the Ace of spades, the Ace of hearts, the 2 of spades and the two of hearts. \((1\spadesuit, 1\heartsuit, 2\spadesuit, 2\heartsuit)\)

I choose 2 cards from this deck at random.

Question 1: What is the probability that I have both aces?

Now, suppose I tell you that I have an ace.

Question 2: What is the probability that I have both aces?

Now again, suppose I tell you that I have the ace of spades \((1\spadesuit)\).

Question 3: What is the probability that I have both aces?
LAW OF TOTAL PROBABILITY

- \( S = \cup_{i=1}^{n} B_i \)
- \( B_i \cap B_j = \emptyset, \quad i \neq j \)
- \( A = \cup_{i=1}^{n} AB_i \)
- \( P(A) = \sum_{i=1}^{n} P(AB_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i) \)

"PROOF":

\[
P(A) = P(A \cap S) = P(A \cap (\bigcup_{i} B_i)) = P(\bigcup_{i} (A \cap B_i))
\]

\[
= \Sigma_i P(A \cap B_i) = \Sigma_i P(B_i)P(A \mid B_i)
\]

AXIOM 3 \quad MULTIPLICATION RULE
**Interpretation**

$P(B)$ is a “prior” probability.

$P(B|A)$ is the “posterior probability”

\[ P(B|A) = \frac{P(B)P(A|B)}{P(A)} = \frac{P(B)P(A|B_k)}{\sum P(A|B_k)P(B_k)} \]

\[ = \frac{P(B|A)}{P(A)} \]
A BAYES' RULE PROBLEM

We wish to understand something about the detection of HIV in the population. Good tests exist but no test is totally reliable. We want to know that if an individual tests positive, what the probability is that individual actually has the disease.

- \( D = \{ \text{disease is present} \} \),
- \( T^+ = \{ \text{test for the disease is positive} \} \)
- \( \bar{D} = \{ \text{disease is not present} \} \)
- \( P(\text{Test says D when D}) = P(T^+|D) = 0.995 \)
- \( P(\text{Test says D when it is not}) = P(T^+|\bar{D}) = 0.01 \)
- \( P(\text{Disease is present}) = P(D) = 0.001 \)

- \( P(D|T^+) = \frac{P(T^+\cap D)}{P(T^+)} \)

- \( \frac{P(T^+|D)P(D)}{P(T^+|D)P(D) + P(T^+|\bar{D})P(\bar{D})} = 0.09058 \)
AN EXERCISE

Computer analysis firm.
Programs 80% in C, 20% in FORTRAN.
Probability of successful compilation on the first run:
    0.2 in C, 0.6 if in FORTRAN

If a randomly selected program compiles on the first run, what is the probability that it was written in C?
Bayes' Rule: A Generalization

Supplementary Exercise # 25

\[ S = \bigcup_{c \in C} A_c, \quad A_i \cap A_j = \emptyset \quad \forall i, j \]

\[ P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)} = \frac{\sum_{c \in C} P(B|A_c)P(A_c)}{\sum_{c \in C} P(B|A_c)P(A_c)} \]

- The Bayesian Paradigm
  - Prior distribution
  - Posterior distribution
  - Evidence

- Example: Software engineering project, 33 'programs'
  - 5 programmers,
  - \( B = \{ \text{program fails} \} \)
  - \( P(A_1) = \frac{12}{33}, \quad P(A_2) = \frac{9}{33}, \quad P(A_3) = \frac{10}{33}, \quad P(A_4) = \frac{5}{33}, \quad P(A_5) = \frac{2}{33} \)
  - \( P(B|A_1) = 0.02, \quad P(B|A_2) = 0.25, \quad P(B|A_3) = 0.05 \)
  - \( P(B|A_4) = 0.15, \quad P(B|A_5) = 0.45 \)

\[ P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{c \in C} P(B|A_c)P(A_c)} \quad i = 1, \ldots, 5 \]

Prior distribution

<table>
<thead>
<tr>
<th>( A_i )</th>
<th>0.364</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_2 )</td>
<td>0.121</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>0.303</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>0.152</td>
</tr>
<tr>
<td>( A_5 )</td>
<td>0.061</td>
</tr>
</tbody>
</table>

Posterior distribution

<table>
<thead>
<tr>
<th>( A_i )</th>
<th>0.071</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_2 )</td>
<td>0.295</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>0.147</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>0.221</td>
</tr>
<tr>
<td>( A_5 )</td>
<td>0.265</td>
</tr>
</tbody>
</table>

1.001
INDEPENDENT EVENTS

From the definition of conditional probability:

\[ P(A|B) = \frac{P(AB)}{P(B)} \quad P(AB) = P(A|B)P(B) \]

Two events A, B are (statistically) independent if and only if

\[ P(A|B) = P(A) \]

(Event B has no influence on event A).

Therefore if events A and B are independent

\[ P(AB) = P(A|B)P(B) = P(A)P(B) \]

Also: A and B are independent events if \( P(AB) = P(A)P(B) \)

SUBLETSY OF INDEPENDENCE

The events A, B, C are independent if and only if

\[ P(ABC) = P(A)P(B)P(C) \]

and they are pairwise independent that is:

\[ P(AB) = P(A)P(B), \]
\[ P(AC) = P(A)P(C), \quad P(BC) = P(B)P(C) \]
Example 5 \hspace{1cm} p.367

Suppose $E$ is the event that a randomly generated bit string of length four begins with a 1 and $F$ is the event that a randomly generated bit string contains an even number of 0s. Are $E$ and $F$ independent, if the 16 bit strings of length four are equally likely?

**Solution:** There are eight bit strings of length four that begin with a 1: 1000, 1001, 1010, 1011, 1100, 1101, 1110, and 1111. There are also eight bit strings of length four that contain an even number of 1s: 0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111. Since there are 16 bit strings of length four, it follows that

$$p(E) = p(F) = \frac{8}{16} = \frac{1}{2}.$$  

Because $E \cap F = \{1111, 1100, 1010, 1001\}$, we see that

$$p(E \cap F) = \frac{4}{16} = \frac{1}{4}.$$  

Since

$$p(E \cap F) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = p(E)p(F),$$

we conclude that $E$ and $F$ are independent.

---

Example 6

Assume, as in Example 4, that each of the four ways a family can have two children is equally likely. Are the events $E$, that a family with two children has two boys, and $F$, that a family with two children has at least one boy, independent?

**Solution:** Since $E = \{BB\}$, we have $p(E) = \frac{1}{4}$. In Example 4 we showed that $p(F) = \frac{3}{4}$ and that $p(E \cap F) = \frac{1}{4}$. Since $p(E \cap F) = \frac{1}{4} \neq \frac{3}{16} = \frac{1}{4} \cdot \frac{3}{4} = p(E)p(F)$, events $E$ and $F$ are not independent.

Example 7

Are the events $E$, that a family with three children has children of both sexes, and $F$, that a family with three children has at most one boy, independent? Assume that the eight ways a family can have three children are equally likely.
Example 7 \hspace{1cm} P368

\[ S = \{ \text{children family} \} \]

\[ p(B) = p(G) = \frac{1}{2} \text{ on each birth} \implies |S| = 2^3 = 8 \]

\[ \{ BBB, BBG, BGB, GBB, GGB, GBG, BGG, GGG \} \]

\[ \{ w_1, w_2, \ldots, w_8 \} \quad P(w_c) = \frac{1}{8}, \ c = 1, 2, \ldots, 8 \]

\[ E = \{ \text{there are children of both sexes} \} \]

\[ = \{ BBC, BGB, GBB, GGB, GBC, GGB, BCG \} \]

\[ |E| = 6 \]

\[ F = \{ \text{at most one boy} \} \]

\[ = \{ GCB, GBC, BCG, GGG \} \]

\[ |F| = 4 \]

Question: (Q) Are E and F independent?

\[ P(EB) = \frac{3}{8} \]

\[ = \frac{P(E)P(F)}{?} \]

\[ = \frac{6}{8} \times \frac{4}{8} = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8} \]

Answer: (A) Yes

Exercise 5.2:27 \hspace{1cm} P377

\[ S = \{ \text{n children family} \}, \quad |S| = 2^n \]

E, F, C \hspace{0.5cm} \text{as above}

\[ \text{Q} \hspace{0.5cm} \text{are E, F independent?} \]

a) \( n = 2 \hspace{1cm} \text{No} \)

b) \( n = 4 \hspace{1cm} \text{No} \)

c) \( n = 5 \hspace{1cm} \text{No} \)
Independent Events, Examples

Example 1. Let two dice be tossed. Let $S = \{(i, j), i, j \in \{1, 2, \ldots, 6\}\}$, $S = \{\omega_1, \omega_2, \ldots \omega_{36}\}$, $|S| = 36$, $p(\omega_i = 1/36, i = 1, 2, \ldots 36)$

Let

- $A = \{\text{first die = 1, 2, or 3}\}$
- $B = \{\text{first die = 3, 4, or 5}\}$
- $C = \{\text{the sum of the two faces is 9}\}$

(Thus $A \cap B = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}$, $A \cap C = \{(3, 6)\}$, $B \cap C = \{(3, 6), (4, 5), (5, 4)\}$, $A \cap B \cap C = \{(3, 6)\}$.)

Then

\[
P(A \cap B) = 1/6 \neq P(A)P(B) = 1/2(1/2) = 1/4 \\
P(A \cap C) = 1/36 \neq P(A)P(C) = 1/2(4/36) = 1/18 \\
P(B \cap C) = 1/12 \neq P(B)P(C) = 1/2(1/9) = 1/18
\]

But

\[
P(A \cap B \cap C) = 1/36 = P(A)P(B)P(C)
\]

Example 2. Now in the same probability space let

- $A = \{\text{first die = 1, 2, or 3}\}$
- $B = \{\text{second die = 4, 5, or 6}\}$
- $C = \{\text{the sum of the two faces is 7}\}$

(Thus $\{A \cap C\} = \{(1, 6), (2, 5), (3, 4)\} = A \cap C$ etc.) Then

\[
P(A \cap B) = 1/4 = P(A)P(B) = 1/2(1/2) \\
P(A \cap C) = 1/12 = P(A)P(C) = 1/2(1/6) \\
P(B \cap C) = 1/12 = P(B)P(C) = 1/2(1/6)
\]

But

\[
P(A \cap B \cap C) = 1/12 \neq P(A)P(B)P(C) = 1/24
\]
CONDITIONAL PROBABILITY AND INDEPENDENT EVENTS

Consider the experiment of rolling three unbiased three-sided dice. (Each die will show one of the numbers 1, 2 or 3.) Compute the probability that:

a) Each face is odd.

b) The sum of the faces exceeds 6 when there is at least one "1".

c) The sum is prime when exactly two faces are the same.
CONDITIONAL PROBABILITY PROBLEMS

Consider a room with 100 people of whom 60 are men, 30 are young, 10 are young men, 40 are Republicans, 20 are Republican men, 15 are young Republicans and 5 are young Republican men. Suppose one person is chosen at random and discovered to be a man. What is the probability that the person is an old Republican?

In a certain city, registered Democrats outnumber registered Republicans by 4 to 1. In the last election, 90% of registered Republicans voted Republican and 60% of registered Democrats voted Democratic. If a voter is known to have voted Republican, find the probability that she/he is a registered Democrat.
Exercises 5.2

1. What probability should be assigned to the outcome of heads when a biased coin is tossed, if heads is three times as likely to come up as tails? What probability should be assigned to the outcome of tails?

3. Find the probability of each outcome when a biased die is rolled, if rolling a 2 or rolling a 4 is three times as likely as rolling each of the other four numbers on the die and it is equally likely to roll a 2 or a 4.
23. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

25. What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive 0s, given that the first bit is a 1? (Assume the probabilities of a 0 and a 1 are the same.)
27. Let $E$ and $F$ be the events that a family of $n$ children has children of both sexes and has at most one boy, respectively. Are $E$ and $F$ independent if

- a) $n = 2$?
- b) $n = 4$?
- c) $n = 5$?

29. A group of six people play the game of "odd person out" to determine who will buy refreshments. Each person flips a fair coin. If there is a person whose outcome is not the same as that of any other member of the group, this person has to buy the refreshments. What is the probability that there is an odd person out after the coins are flipped once?