Problem 5.2.1. Let $X_n$ denote the mean of a random sample of size $n$ from a distribution that is $N(\mu, \sigma^2)$. Find the limiting distribution of $X_n$.

Solution 5.2.1. Since the random sample is taken from a distribution with finite mean $\mu$ and finite variance $\sigma^2$, we may apply the weak law of large numbers to conclude that $\{X_n\}$ converges to $\mu$ in probability. Theorem 5.2.1 says that if a sequence of random variables $\{Y_n\}$ converges to the random variable $Y$ in probability, then $\{Y_n\}$ converges to $Y$ in distribution. Therefore, since $\{X_n\}$ converges to $\mu$ in probability we conclude that $\{X_n\}$ converges to $\mu$ in distribution. Hence the limiting distribution for $X_n$ is the distribution of the degenerate random variable $\mu$, $F_{\mu}(x) = \begin{cases} 0, & x < \mu \\ 1, & x \geq \mu. \end{cases}$

Problem 5.2.2. Let $Y_1$ denote the minimum of a random sample of size $n$ from a distribution that has pdf $f(x) = e^{-(x-\theta)}$, $\theta < x < \infty$, zero elsewhere. Let $Z_n = n(Y_1 - \theta)$. Investigate the limiting distribution of $Z_n$.

Solution 5.2.2. A routine calculation shows that $P(Y_1 > y) = e^{-n(y-\theta)}$ for $y > \theta$, and $P(Y_1 > y) = 1$ for $y \leq \theta$. Since the support of $Y_1$ is the interval $(\theta, \infty)$ it follows that the support of $Z_n = n(Y_1 - \theta)$ is the interval $(0, \infty)$ and therefore $F_{Z_n}(t) = P(Z_n \leq t) = 0$ for all $t \leq 0$. Let $t > 0$, then

$$F_{Z_n}(t) = P(Z_n \leq t) = P(n(Y_1 - \theta) \leq t) = P(Y_1 \leq \frac{t}{n} + \theta) = 1 - P(Y_1 > \frac{t}{n} + \theta) = 1 - e^{-t}.$$ We see that, for every natural number $n$, $F_{Z_n}$ is the cdf for the exponential distribution with mean $\mu = 1$. Therefore, $\{Z_n\}$ converges in distribution to an exponential distribution with mean $\mu = 1$.

Problem 5.2.5. Let the pmf of $Y_n$ be $p_n(y) = 1$, $y = n$, zero elsewhere. Show that $Y_n$ does not have a limiting distribution. (In this case, the probability has “escaped” to infinity.)

Solution 5.2.5. The cdf for the degenerate random variable $Y_n$ is

$$F_{Y_n}(y) = \begin{cases} 0, & y < n \\ 1, & y \geq n. \end{cases}$$ For all $y$, $\lim_{n \to \infty} F_{Y_n}(y) = 0$. To show that $Y_n$ does not have a limiting distribution, we must show that there does not exist a distribution function
$F$ with the property that $F(y) = 0$ for every $y \in C(F)$. Equivalently, we must show that if $F$ is a distribution function then there exists a point $y$ such that $F$ is continuous at $y$ and $F(y) \neq 0$.

Let $F$ be a distribution function. Since $\lim_{y \to \infty} F(y) = 1$, there exists a real number $y_0$ such that $F(y) > 1/2$ for all $y > y_0$. Recall that the set of discontinuous points for a distribution is always a countable set. Since the interval $(y_0, \infty)$ is an uncountable set, there must be at least one point $y \in (y_0, \infty)$ such that $F$ is continuous at $y$. Since $y > y_0$, we also have that $F(y) > 1/2 > 0$. Therefore every distribution function $F$ has at least one point $y$ such that $F$ is continuous at $y$ and $F(y) \neq 0$. We conclude that $Y_n$ does not have a limiting distribution.

**Problem 5.2.7.** Let $X_n$ have a gamma distribution with parameter $\alpha = n$ and $\beta$, where $\beta$ is not a function of $n$. Let $Y_n = X_n/n$. Find the limiting distribution of $Y_n$. 

**Solution 5.2.7.** From Appendix D on page 667 we see that $E[X_n] = n\beta$ and $\text{Var}[X_n] = n\beta^2$, and therefore $E[Y_n] = E[X_n/n] = \beta$ and $\text{Var}[Y_n] = \text{Var}[X_n/n] = \beta^2/n$. Since the expected value of $Y_n$ does not depend on $n$ and since the variance of $Y_n$ is a constant divided by $n$, Exercise 5.1.3 on page 293 shows that $\{Y_n\}$ converges to $\beta$ in probability. Applying Theorem 5.2.1 on page 298 shows that $\{Y_n\}$ converges to $\beta$ in distribution. We conclude that the limiting distribution of $Y_n$ is the distribution for the degenerate random variable $\beta$,

$$F_\beta(t) = \begin{cases} 0, & t < \beta \\ 1, & t \geq \beta \end{cases}$$

**Problem 5.2.12.** Prove Theorem 5.2.3.

**Solution 5.2.3.**

**Theorem.** Suppose $X_n$ converges to $X$ in distribution and $Y_n$ converges in probability to $0$. Then $X_n + Y_n$ converges to $X$ in distribution.

**Proof.** Let $x$ be a continuous point for $F_X$. We want to show that $\lim_{n \to \infty} F_{X_n+Y_n}(x) = F_X(x)$.

Let $\epsilon > 0$ be given. Suppose that $x + \epsilon$ and $x - \epsilon$ are also continuous points for $F_X$. Observe that $X_n + Y_n \leq x$ and $|Y_n| \leq \epsilon$ imply that $X_n \leq x + \epsilon$. Thus,

$$F_{X_n+Y_n}(x) = P(X_n + Y_n \leq x)$$
$$= P(X_n + Y_n \leq x, |Y_n| < \epsilon) + P(X_n + Y_n \leq x, |Y_n| \geq \epsilon)$$

$$\leq P(X_n + Y_n \leq x, |Y_n| \leq \epsilon) + P(X_n + Y_n \leq x, |Y_n| \geq \epsilon)$$

$$\leq P(X_n \leq x + \epsilon) + P(|Y_n| \geq \epsilon)$$

$$= F_X(x + \epsilon) + P(|Y_n| \geq \epsilon).$$
Thus,
\[ \limsup_{n \to \infty} F_{X_n}(x + \epsilon) = F_X(x + \epsilon), \]
and since \( Y_n \xrightarrow{P} 0 \) we have that
\[ \lim_{n \to \infty} P(|Y_n| \geq \epsilon) = 0. \]
The inequality in (1), together with these last two limits, shows that
\[ (2) \quad \limsup_{n \to \infty} F_{X_n+Y_n}(x) \leq F_X(x + \epsilon). \]

Now, observe that \( X_n + Y_n > x \) and \( |Y_n| < \epsilon \) imply that \( X_n > x - \epsilon \).
Thus,
\[ 1 - F_{X_n+Y_n}(x) = P(X_n + Y_n > x) \]
\[ = P(X_n + Y_n > x, |Y_n| < \epsilon) + P(X_n + Y_n > x, |Y_n| \geq \epsilon) \]
\[ \leq P(X_n > x - \epsilon) + P(|Y_n| \geq \epsilon) \]
\[ = 1 - F_{X_n}(x + \epsilon) + P(|Y_n| \geq \epsilon). \]
Rearranging the terms of this inequality yields
\[ (3) \quad F_{X_n}(x - \epsilon) - P(|Y_n| \geq \epsilon) \leq F_{X_n+Y_n}(x). \]

Since \( X_n \xrightarrow{D} X \) and since \( x - \epsilon \) is a continuity point for \( F_X \) we have that
\[ \lim_{n \to \infty} F_{X_n}(x - \epsilon) = F_X(x - \epsilon), \]
and since \( Y_n \xrightarrow{P} 0 \) we have that
\[ \lim_{n \to \infty} P(|Y_n| \geq \epsilon) = 0. \]
The inequality in (3), together with these last two limits, shows that
\[ (4) \quad F_X(x - \epsilon) \leq \liminf_{n \to \infty} F_{X_n+Y_n}(x). \]
Combining inequalities (2) and (4) we have
\[ (5) \quad F_X(x - \epsilon) \leq \liminf_{n \to \infty} F_{X_n+Y_n}(x) \leq \limsup_{n \to \infty} F_{X_n+Y_n}(x) \leq F_X(x + \epsilon). \]

The inequalities in (5) require that \( x - \epsilon \) and \( x + \epsilon \) be continuity points for \( F_X \). We claim that there exists a sequence of positive numbers \( \{\epsilon_k\} \) such that \( \lim_{k \to \infty} \epsilon_k = 0 \) and for all natural numbers \( k \) the points \( x \pm \epsilon_k \) are continuity points for \( F_X \). This claim is an immediate consequence of the fact that the set of discontinuous points of \( F_X \) is countable.

Choose \( \{\epsilon_k\} \) as described in the previous paragraph. Since \( F_X \) is continuous at \( x \), \( \lim_{k \to \infty} F_X(x \pm \epsilon_k) = F_X(x) \). Since \( F_X \) is continuous at the points \( x \pm \epsilon_k \) for every natural number \( k \),
\[ (6) \quad F_X(x - \epsilon_k) \leq \liminf_{n \to \infty} F_{X_n+Y_n}(x) \leq \limsup_{n \to \infty} F_{X_n+Y_n}(x) \leq F_X(x + \epsilon_k). \]
Finally, letting \( k \to \infty \) in (7) shows that \( \lim_{n \to \infty} F_{X_n+Y_n}(x) = F_X(x) \), and we conclude that \( X_n + Y_n \xrightarrow{D} X \). \( \square \)
**Problem 5.2.20.** Use Stirling’s formula, (5.2.2), to show that the first limit in Example 5.2.3 is 1.

**Solution 5.2.20.** Let us introduce three sequences,

\[
a_n = \frac{\Gamma \left( \frac{n-1}{2} + 1 \right)}{\sqrt{2\pi} \left( \frac{n-1}{2} \right)^{n/2} e^{-(n-1)/2}}
\]

(7)

\[
b_n = \frac{\sqrt{2\pi} \left( \frac{n-2}{2} \right)^{(n-1)/2} e^{-(n-2)/2}}{\Gamma \left( \frac{n-2}{2} + 1 \right)}
\]

(8)

\[
c_n = e^{-1/2} \sqrt{1 - \frac{2}{n} \left( \frac{n-1}{n-2} \right)^{n/2}}.
\]

(9)

Stirling’s formula states that

\[
\lim_{k \to \infty} \frac{\Gamma(k+1)}{\sqrt{2\pi k^{k+1/2} e^{-k}}} = 1.
\]

We may apply Stirling’s formula to \(a_n\) with \(k = (n-1)/2\) to conclude that \(\lim_{n \to \infty} a_n = 1\). In a similar way we may apply Stirling’s formula to \(b_n\) with \(k = (n-2)/2\) to conclude that \(\lim_{n \to \infty} b_n = 1\).

Notice that \(\lim_{n \to \infty} \left[\frac{(n-1)/(n-2)}{n/2}\right]^{n/2}\) is a \(1^\infty\) indeterminate form. Applying Calculus II methods that are taken from the section on L'Hôpital’s Rule we are able to evaluate that limit. We leave it as an exercise to confirm that \(\lim_{n \to \infty} \left[\frac{(n-1)/(n-2)}{n/2}\right]^{n/2} = e^{1/2}\). Knowing this, we see that \(\lim_{n \to \infty} c_n = 1\).

After some simplification, which we leave to the student, we find that

\[
a_n \cdot b_n \cdot c_n = \frac{\Gamma((n+1)/2)}{\sqrt{n/2} \Gamma(n/2)}.
\]

We therefore conclude that

\[
\lim_{n \to \infty} \frac{\Gamma((n+1)/2)}{\sqrt{n/2} \Gamma(n/2)} = \lim_{n \to \infty} a_n \cdot b_n \cdot c_n = 1 \cdot 1 \cdot 1 = 1.
\]