Problem 1. [15 Points] Moon Market sells 12,000 bags of Emerald T rice per year at a steady rate. The manager of the store places several orders of the same size spaced equally throughout the year. It costs $160.00 to place an order and get it delivered. The carrying cost is $6.00 per bag and year.

1. What is the inventory cost if the manager places one order per month.

2. How many bags of rice should the manager order each time to minimize the inventory cost, and what is the inventory cost in this case?

Solution: (Compare Problem 5 in Section 2.6) Order $x$ bags of rice at a time placing $12,000/x$ orders. The cost will be

$$C(x) = 3x + \frac{12,000}{x} \cdot 160$$

1.) If we place one order a month, ordering 1,000 bags, then the cost will be

$$C(1000) = 3000 + 12 \cdot 160 = 4,920 \text{ (dollars)}.$$ 

2.) To minimize the inventory cost we find the critical points of $C(x)$.

$$C'(x) = 3 - \frac{12,000}{x^2} \cdot 160$$

and $C'(x) = 0$ if and only if $x^2 = 4,000 \cdot 160$, or $x = 800$.

Obviously $C''(800) > 0$, so that we have a local minimum at $x = 800$. Being the only critical point, it is also the absolute minimum for $x > 0$. In this case the inventory cost will be

$$C(800) = 2,400 + 15 \cdot 160 = 4,800 \text{ (dollars)}.$$ 

Problem 2. [15 Points] Differentiate the following functions:

a.) $f(x) = e^{\frac{x^2+3}{2}}$  

b.) $g(x) = 5^{2x}$ 

c.) $h(x) = x \ln(2 + x^2)$

Solution: (The problems are similar to # 13 in Section 4.3, # 15 in Section 4.5. For b.) see the discussion at the end of Section 4.3.)

$$f'(x) = \left(\frac{x^2+3}{2}\right) e^{\frac{x^2+3}{2}} = xe^{\frac{x^2+3}{2}}$$

$$g'(x) = (e^{2x\ln5})' = (2\ln5)^{2x}$$

$$h'(x) = \ln(2 + x^2) + x \frac{2x}{2 + x^2} = \ln(2 + x^2) + \frac{2x^2}{2 + x^2}$$
Problem 3. [15 Points] A PC manufacturer estimates that $t$ months from now it will sell $x$ thousand of its model X units per month, where $x = .05t^2 + 2t + 5$. The profit from manufacturing and selling $x$ thousand units is estimated to be $P = .001x^2 + .1x - .25$ million dollars. Calculate the rate at which the profit will be increasing 5 months from now.

Solution: (For a slightly differently worded problems with the same numbers see # 56 in Section 3.2.) A bare handed solution is:

$$P(t) = .001x^2(t) + .1x(t) - .25$$
$$= .001(.05t^2 + 2t + 5)^2 + .1(.05t^2 + 2t + 5) - .25.$$

Then

$$P'(t) = .001 \cdot 2(.05t^2 + 2t + 5)(.1t + 2) + .1(.1t + 2).$$

and

$$P'(5) = .001 \cdot 2 \cdot 16.25 \cdot 2.5 + .1 \cdot 2.5$$
$$= .08125 + .25$$
$$= .33125.$$

We found that the profit will be increasing at a rate of $331250$ per month.

More elegantly and clearly, we can use the formula for related rates:

$$\frac{dP}{dt} = \frac{dP}{dx} \frac{dx}{dt}$$
and

$$\left. \frac{dP}{dt} \right|_{t=5} = \left. \frac{dP}{dx} \right|_{x=x(5)} \cdot \left. \frac{dx}{dt} \right|_{t=5}.$$

We compute the individual terms. At time $t = 5$ we find $x = 16.25$. Furthermore

$$\frac{dP}{dx} = .002x + .1 \quad \text{and} \quad \left. \frac{dP}{dx} \right|_{x=16.25} = .1325$$
$$\frac{dx}{dt} = .1t + 2 \quad \text{and} \quad \left. \frac{dx}{dt} \right|_{t=5} = 2.5$$

Taken together

$$\left. \frac{dP}{dt} \right|_{t=5} = \left. \frac{dP}{dx} \right|_{16.25} \cdot \left. \frac{dx}{dt} \right|_{t=5} = .1325 \cdot 2.5 = .33125,$$

yielding the same answer as above.
Problem 4. [15 Points] Suppose that the price $p$ (in dollars) and the weekly sales of $x$ cases of purple papaya satisfy the demand equation $2p^3 + x^2 = 4500$. Determine the rate at which sales are changing at a time when $x = 50$, $p = 10$, and the price is falling at a rate of $.50$ per week.

Solution: (For a slightly differently worded problems with the same numbers see # 39 in Section 3.3. The numbers are changed a slightly as compared to the exam) We differentiate the demand equation with respect to the variable $t$, resulting in:

$$6p^2 \frac{dp}{dt} + 2x \frac{dx}{dt} = 0.$$ 

We are supposed to find $\frac{dx}{dt}$ at a certain moment, so we solve for it:

$$\frac{dx}{dt} = -\frac{3p^2}{x} \frac{dp}{dt}.$$ 

We are given $x = 50$, $p = 10$, and $\frac{dp}{dt} = -.5$. We calculate that $\frac{dx}{dt} = 3$, and conclude that sales will increase at a rate of 3 cases of purple papaya.

Problem 5. [15 Points] Radioactive Cobalt 60 has a halflife of 5.3 years.

1. What is the decay rate $\alpha$ of Cobalt 60?

2. How long does it take to until 90% of a sample of Cobalt 60 has decayed?

Solution: If we denote the decay rate $\alpha$ then we learned that $\alpha$ times the halflife is ln 2. We conclude that

$$\alpha = \frac{\ln 2}{5.3} \approx .13.$$ 

If $Y_0$ is the initial amount of the sample (at time $t = 0$), then the amount is $Y(t) = Y_0 e^{-\alpha t}$ at time $t$. You are suppose to find $t$ so that

$$Y(t) = .1Y_0 = Y_0 e^{-\alpha t} \quad \text{or} \quad .1 = e^{-\alpha t}.$$ 

After applying $\ln$ to both sides of the equation we find

$$-\ln(10) = \ln(.1) = -\alpha t \quad \text{and} \quad t = \frac{5.3 \ln(10)}{\ln(2)} \approx 17.27 \text{ (years)}.$$
Problem 6. [15 Points] Suppose that you are saving at a rate of $1,000.00 per month, that the bank pays interest at an annual rate to 6%, and that you start out with a balance of $5,000.00. Denote by $A(t)$ the account balance $t$ years from now. Set up the initial value problem for $A(t)$.

Solution: The initial value problem is

$$A'(t) = .06A(t) + 12,000$$
$$A(0) = 5,000.$$

The first equation tells us how $A(t)$ changes. We add interest and principle. The second equation tells us the balance at time $y = 0$.

Problem 7. [15 Points] Currently 1,800 people ride the A-train each day and pay $4.00 for a ticket. The number of people $q$ willing to ride the train at price $p$ is $q = 600(5 - \sqrt{p})$. The railroad would like to increase the revenue.

1. Is the demand elastic or inelastic at $p = 4$.

2. Should the price be raised or lowered to increase the revenue?

Solution: (For a problem with the same numbers and slightly different wording see # 22 in Section 5.3) Denote the demand function by $f(p) = q$. Then $f(4) = 1800$. Also $f'(p) = -300/\sqrt{p}$ and $f'(4) = -150$. For the elasticity we calculate:

$$E(p) = \frac{pf'(p)}{f(p)} = \frac{300p}{600\sqrt{p}(5 - \sqrt{p})} = \frac{\sqrt{p}}{2(5 - \sqrt{p})} \quad \text{and} \quad E(4) = \frac{1}{3}.$$ 

Because $E(4) < 1$, the demand is inelastic at the price of $p = 4$, and the price should be raised to increase revenues.