Find the derivative of each of these, but DO NOT SIMPLIFY. Calculators are not allowed. Passing is 5 out of 6. Passed exams will be given to your instructor. Allowed time: 15 minutes.

1. \( y = (x^2 + 1)^{-3/5} \)
   \[
y' = -\frac{3}{5} (x^2 + 1)^{-8/5} - 2x
   \]

2. \( y = (x^2)^{-8/9} \)
   \[
y' = -\frac{16}{q} x^{-40/9}
   \]

3. \( y = \cot(x) \sin(2x - 1) \)
   \[
y' = -\csc^2(x) \sin(2x-1) + 2 \cot(x) \cos(2x-1)
   \]

4. \( f(x) = \frac{2}{3} \cos(x^{-1/4} - 2x) \)
   \[
f' = -\frac{2}{3} \sin \left( x^{-1/4} - 2x \right) \left( -\frac{1}{4} x^{-5/4} - 2 \right)
   \]

5. \( y = \frac{x^{3/7}}{x^2 - 3x - 1} \)
   \[
y' = \frac{\frac{3}{7} x^{-4/7} (x^2 - 3x - 1) - x^{3/7} (2x - 3)}{(x^2 - 3x - 1)^2}
   \]

6. \( h(t) = t^5(t^2 - 2)^3 + 0.8(27t^{0.15}) - t \)
   \[
h' = 5t^4(t^2 - 2)^3 + 3t^3(2t^2 - 2) + 0.8 \cdot 0.15(27t)^{0.14} \cdot 27 - 1
   \]
Calculus I (Math 241) – Test 1

Problem 1. [10 Points] Find the equation of the tangent line to the graph of the function \( f(x) = x^2 \sin x \) at \( x = \pi/3 \).

\[
\begin{align*}
\frac{d}{dx}\left(\frac{\pi}{3}\right) &= \frac{\pi^2}{9} \frac{\sqrt{3}}{2} = \frac{\pi^2}{18} \\
\frac{d}{dx}(x) &= 2x \sin x + x^2 \cos x \\
\frac{d}{dx}\left(\frac{\pi}{3}\right) &= \frac{2\pi}{3} \frac{\sqrt{3}}{2} + \frac{\pi^2}{9} \frac{1}{2} = \frac{\pi\sqrt{3}}{3} + \frac{\pi^2}{18} \\
\end{align*}
\]

\[ t(x) = \left(\frac{\pi\sqrt{3}}{3} + \frac{\pi^2}{18}\right)(x - \frac{\pi}{3}) + \frac{\pi \sqrt{3}}{18} \quad \text{(Equation of tangent line)} \]

Problem 2. [5 Points] Spell out in exact mathematical terms the meaning of the statement

\[ \lim_{x \to a} H(x) = L, \]

assuming that the function \( H(x) \) is defined on an open interval that contains \( a \), but possibly not at \( a \) itself.

For all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( 0 < |x - a| < \delta \) implies that \( |H(x) - L| < \varepsilon \).

Problem 3. [10 Points] Work out the limit of the difference quotient

\[ \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \]

for \( a > 0 \).

\[
\begin{align*}
\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} &= \lim_{x \to a} \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{(x - a)(\sqrt{x} + \sqrt{a})} \\
&= \lim_{x \to a} \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} \\
&= \frac{1}{2 \sqrt{a}}
\end{align*}
\]

Suppose \( f(a) = A \) and \( f(b) = B \), and \( f \) is defined and continuous on the closed interval with end points \( a \) and \( b \). If \( C \) is in between \( A \) and \( B \), then there exists some \( c \) between \( a \) and \( b \) such that \( f(c) = C \).

Problem 5. [15 Points] Consider the polynomial \( p(x) = x^3 - 9x^2 + 23x - 14 \).

1. Show that \( p(x) \) has a zero between 0 and 1.

2. Use \( x = 1 \) as a first guess for the zero, and apply Newton's method to improve the guess once.

\[
p(0) = -14 \quad p(1) = 1 - 9 + 23 - 14 = 1
\]

As a polynomial, \( p(x) \) is continuous on \([0, 1]\), \( p(0) < 0 \) and \( p(1) > 0 \). According to the Intermediate Value Theorem, \( f(c) = 0 \) for some \( c \) between 0 and 1.

\[
p'(x) = 3x^2 - 18x + 23 \quad p'(1) = 8 \quad x_0 = 1
\]

\[
x_1 = x_0 - \frac{p(x_0)}{p'(x_0)} = 1 - \frac{1}{8} = 7/8 \quad \text{(This is the improved guess.)}
\]
Problem 6. [15 Points] Consider the curve given by the equation

\[ x^2 y + xy^3 - 6 = 0. \]

1. Find \( dy/dx \) by implicit differentiation.

2. Note that \((2, 1)\) is a point on the curve. Use approximation by differentials to find an approximate value for \( y(2.3) \).

\[ (x^2 + 3xy^2) \frac{dy}{dx} = - \left[ 2xy + y^3 \right] \]

\[ \frac{dy}{dx} = - \frac{2xy + y^3}{x^2 + 3xy^2} \]

\[ \left. \frac{dy}{dx} \right|_{(x,y)=(2,1)} = - \frac{4 + 1}{4 + 6} = -\frac{5}{10} = -\frac{1}{2} \]

Check: \( 2 \cdot 1 + 2 \cdot 1^3 - 6 = 0 \)

\[ y(2.3) \approx y(2) + y'(2) (2.3 - 2) \]

\[ = 1 - \frac{1}{2} (0.3) \]

\[ = 0.85 \]
Problem 7. [10 Points] A plane flying horizontally at an altitude of 3000 meters (3 km) and a speed of 800 km/h passes directly over Diamond Head. Find the rate at which the distance from the plane to Diamond Head is increasing when it is 4 km away from Diamond Head.

\[ D(t) \rightarrow \text{horizontal distance } x(t) \]
\[ 3000 \text{m} = 3 \text{km} \]
\[ \text{actual distance } D(t) \]

\[ \frac{dD}{dt} = \frac{dD}{dx} \cdot \frac{dx}{dt} \]
\[ = \frac{x}{\sqrt{x^2 + 9}} \cdot 800 \]
\[ = \frac{17}{4} \cdot 800 \]
\[ = 200 \sqrt{17} \]
\[ \approx 530 \]

At the given moment, the distance increases at a rate of 530 km/h.