Lab Suggestions for Calculus I, Math 241/251

**Getting Started.** Introduce the students to the basics of DfW. Open the program, explain what you see, big white work page, line where you enter expression, pull down menus, tool bar, and special character pads. Next you can start doing things.

Entering expressions and operations:

1. 3 + 5, enter, then simplify.

2. 1/3, enter, then show the difference between simplify and approximate.

3. Simplify \((-1)^{1/3}\) and solve \(x^3 = -1\).

4. Enter \(\frac{x^3-a^3}{x-a}\), illustrate the need of parentheses to get exactly this expression. Notice, when you simplify the program performs a long division.

5. Square roots can be entered using the special character pad, or using the function `sqrt`. Even \(\sqrt{x-a}\) and \(\sqrt[1/3]{x-a}\) are simplified by long division.

6. Mention the mathematical constants \(\pi\), the Euler number \(e\), and the factor \(\circ = \pi/180\) for converting degrees into radians on the pad for special characters.

7. Enter the equation \(x^3 - 5x^2 - x + 5 = 0\), and find its solutions algebraically using the command `solve`. You can solve other equations numerically.

8. DfW knows a lot of functions. Just type \(\sin x\), \(\tan x\), and you have the function to work with. The inverse functions are denoted by, e.g., \(\text{asin } x\). You can find a long list of functions by going to the help index under Functions, Derive Functions.

9. You can use the substitute command to replace a variable by anything of your choosing. So, if you substitute a number, then you can evaluate a function at a point.

10. The program will implement some identities when you simplify, e.g., \(\sin^2 x + \cos^2 x\) simplifies to 1. Other simplifications don’t look particularly natural.
Basic graphing:

1. Open a graphing window by clicking on the button with the sine curve on the tool bar. Click on Window, and tile vertically. Then you will have your algebra window and a graphing window.

2. In the algebra window, highlight a function which you like to graph, click on the button with the sine curve at the top of the graphing window. You will see the graph of the function.

3. Use the zoom buttons for zooming in and out, one direction at a time, or both.

4. You can make a specific point the center of the graphing window, click on the point, then the appropriate button on the tool bar.

5. Toggle between the regular and the tracing mode (F3 key or appropriate button on the tool bar), and explore tracing.

6. For a given function, you might use the tracing mode to find the coordinates of a point, the zeros, or extrema.

7. You can graph function, and you can graph the solutions of an equation in two unknowns. Try the hyperbola \( x^2 - y^2 = 1 \), or other equations of your choice.

8. You can graph in the cartesian plan (default), or you can use polar coordinates (Set/Coordinate System/Polar).

Early vectors:

1. Type \([1, 2]\) to get the point with the indicated coordinates. You can graph this point. Choose the point size under Options/Display/Points.

2. The product of two vectors, say \([1, 2][1, 5]\) will give you their dot product, and \(\text{abs}([1, 2])\) will give the length of the vector in question. You can add vectors and multiply them with scalars using the standard notation. If \(u\) and \(v\) are vectors, then \(\text{acos}(uv/(\text{abs}(u)\text{abs}(v)))\) will give you the angle between \(u\) and \(v\).
3. Here is an example of a parametrized curve in the plane. Enter the expression in the algebra window.

\[ [t^3 - 2t, t^2 - t]. \]

Graph it. Problem 40 on page 69 in Stewart’s Early Vectors book gives you more examples to play with, or use any of your favorite curves.

**Difference Quotients, Tangent and Secant Lines.** We use different means to explore the concepts entering in the definition of the derivative.

Pick a function \( f(x) \) and a point \( a \) in its domain. For the purpose of being concrete, let is say \( f(x) = \tan x \) and \( a = \pi/4 \).

Graph \( f(x) \) and \( t(x) = f'(a)(x - a) + f(a) = 2(x - \pi/4) + 1 \), and you will see that the tangent line is close to the graph.

Graph the difference quotient \( \frac{f(x) - f(a)}{x-a} \), and it will be rather apparent that its limit is 2, as \( x \) approaches \( \pi/4 \). To illustrate the definition of the limit, draw horizontal lines \( y = 2 \pm \epsilon \) with your choice of \( \epsilon \). Pick an appropriate \( \delta \), and draw the lines \( x = \pi/4 \pm \delta \). You see, as long as \( x \in (\pi/4 - \delta, \pi/4 + \delta) \) and \( x \neq \pi/4 \) the difference quotient \( \frac{f(x) - f(a)}{x-a} \) is in \((2 - \epsilon, 2 + \epsilon)\).

Enrich the picture with the graph and the tangent line by adding some secant lines. The equation for the secant line through \((a, f(a))\) and \((b, f(b))\) is

\[
s(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a). \]

With \( a = \pi/4 \) fixed, draw a few of these lines for various values of \( b \), approaching \( a \).

We can do the above numerically. To see that the slopes of the secant lines approach 2, the slope of the tangent line, we make a table:

\[
\text{table(} \frac{f(b) - f(a)}{b - a}, b, [\text{comma separated values for } b]\text{)}
\]

Here \( a = \pi/4 \) is fixed, and for \( b \) we provide a list of numbers approaching \( a \). The first column of the table will have the values for \( b \) and the second one the corresponding values for \( \frac{f(b) - f(a)}{b - a} \).

Let us illustrate numerically that the tangent line is close to the graph. First of all, note that

\[
\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| = \left| \frac{f(x) - t(x)}{x - a} \right|.
\]
Let us look at the table

\[
\begin{array}{|c|c|c|}
\hline
x - a, & \left| \frac{f(x) - t(x)}{x - a} \right|, & \left| \frac{f(x) - t(x)}{(x - a)^2} \right|, \\
\hline
\end{array}
\, x, [csv \ for \ x]
\]

with a comma separated list of values for \( x \) approaching \( a \). The first column gives the distance of \( x \) from \( a \), the second one the difference between the function and the tangent line divided by the distance, and the third one the difference between the function and the tangent line divided by the square of the distance. The entries in the second column go to zero, and the ones in the third column are bounded. In one interpretation the third column tells us that, for some number \( A \),

\[
\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| \leq A|x - a|
\]

for \( x \) near \( a \), and certainly \( \frac{f(x) - f(a)}{x - a} \) converges to \( 2 = f'(a) \). The other interpretation is that

\[
|f(x) - t(x)| \leq A|x - a|^2.
\]

If \( |x - a| \) is small, then \( |x - a|^2 \) is very small, and the constant \( A \) is only a scaling factor.

We can use some elementary algebra to differentiate any polynomial. This is not elegant, but easy on DfW. Choose a polynomial \( p(x) \) and a point \( a \). Expand \( p(x) \) in powers of \( (x - a) \). Then the linear part, the summands of degree 0 and 1, provide you with the tangent line. The coefficient of \((x - a)\) is the derivative. To implement this in derive, enter the polynomial. Substitute \( u + a \) for \( x \) and expand. Reverse the substitution, substituting \((x - a)\) for \( u \). Do not expand, otherwise you are back to where you started. You now have \( p(x) \) in powers of \((x - a)\). For example:

\[
p(x) = x^4 - x^3 - 2x^2 + 7x + 2 = (x - 2)^4 + 7(x - 2)^3 + 16(x - 2)^2 + 19(x - 2) + 16.
\]

Consequently, the equation of the tangent line to the graph of \( p(x) \) at \( x = 2 \) is \( t(x) = 19(x - 2) + 16 \), and \( p'(2) = 19 \). As a word of explanation, if \( x \) is close to \( a \), then \( |x - a| \) is small, and higher powers of \( |x - a| \) are much smaller.
Thus $|p(x) - t(x)|$ is small, expressing that the tangent line is close to the graph, and that is one interpretation of differentiability.

For a nice function $f(x)$, the tangent line at a point $x = a$ is close to the graph of $f$ near $a$. So, following the tangent line instead of the graph of the function introduces only small errors. Here are outlines for a few labs:

**Euler’s method to solve differential equations:** Consider the logistic equation

$$y' = ay - by^2,$$

and to be concrete use $a = .1$ and $b = .0001$. Explain how this differential equation arises, natural exponential growth combined with competition. The rate a which a population grows is proportional to its size. Competition occurs when two members of the population meet, and the probability for this to occur is proportional to the square of the size of the population.

We’ll construct the graph of a solution. Say $[t, y]$ is a point on the graph then a next point will be approximately

$$[t + \Delta, y + (ay - by^2)\Delta].$$

Substitute $a = .1, b = .0001, t = u_1, y = u_2$, and use $\Delta = .5$ to get a concrete version (1)' of (1). Starting out with some $u$, say $u = [0, 10]$, you can repeatedly substitute points into (1)' to get a sequence of next points. You get 100 next points by using iterates ((1)', $u$, $[0,10]$, 100). Plot these points to get an idea of the graph of $y$. Play with different initial conditions.

As additional exercise you can study $y' = ay(1 - \frac{b}{a}y)$ as a function of $y$. This function has two zeros, at $y = 0$ and at $y = a/b = 1000$. If $y \in (0, a/b)$, then $y' > 0$, so that $y$ is increasing. If $y > a/b$ or $y < 0$, then $y' < 0$ and $y$ is decreasing. We deduce that there is a stable equilibrium when $y = a/b$, and an unstable one when $y = 0$. The stable equilibrium at $y = a/b$ is called the carrying capacity. The actual solution of the initial value problem, above differential equation and initial condition $y(t_0) = y_0$, is

$$y(t) = \frac{ay_0}{by_0 + (a - by_0)e^{at-t_0}}.$$

Graph this function in addition to the approximate solutions from above. Apparently this function $y(t)$ satisfies the given initial condition, and by hand or using DfW you may check that it satisfies the differential equation.
Orthogonal trajectories: Consider a family of curves, say
\[ x^2 + ay^2 - 1 = 0, \]
for various values of \( a \). Graph a couple of them by substituting values for \( a \).
To get a family use

\[
\text{vector}(x^2 + ay^2 = 1, a, [\text{csv for } a]).
\]

Graph the family of curves.

Use \text{imp}_\text{dif}(x^2 + ay^2 - 1, x, y, 1) to get \( dy/dx = \frac{dx}{dy} \). By hand, or using \text{solve}
on DfW, we get that \( a = 1 - \frac{x^2}{y^2} \). We substitute \( a \) in the formula for \( dy/dx \) and take the negative reciprocal of it. Then we get \( y' = \frac{x}{1-x^2} \) for the orthogonal trajectory. Apply Euler’s method to construct solution curves with various
initial conditions.

Adjust the step length \( \Delta \) to fit the problem. Play with different values of \( \Delta \) to see how you can make the method fail, and how it stabilizes for small values of \( \Delta \).

Newton’s Method: If \( f \) is a nice function and \( x_0 \) is chosen appropriately, then

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

provides a sequence converging to a zero of \( f \). The approach uses zeros of tangent lines to get improved guesses for the zeros of a function. To implement the process we use

\[
(u =) \ [n + 1, x - \frac{f(x)}{f'(x)}, f(x - \frac{f(x)}{f'(x)})]
\]

with \( n = u_1 \) and \( x = u_2 \) as formula for the iteration. To get a sequence of values we type \text{iterates(}iteration formula, starting point, number of steps\text{)} and approximate. The third entry is just to check how small the value of the function has become, indicating that we are getting close to a zero of the function. You can follow the process graphically: Start with a point \( x \), not too far away from a zero of the function, go to the corresponding point on the graph, draw the tangent line through it, and find the zero of the tangent line. This is the improved guess for the zero of the function. The method iterates this process.
Properties of Graphs: Use the graphs of \( f, f', \) and \( f'' \) (several examples, created by the students) to reinforce the relation between the 1st derivative and monotonicity, the second derivative and concavity, the 1st and 2nd derivative test, and the detection of inflection points.

More explicitly, put up two graphs, one belonging to a function \( f \) one to its derivative \( f' \). Have a student come to the screen and decide which graph belongs to \( f \) and which one to \( f' \). Have the students make the connection: On intervals where \( f' > 0 \) the function \( f \) is increasing. On intervals where \( f' < 0 \) the function \( f \) is decreasing.

Do the same with a pair of graphs, one of a function \( f \) and the other one for its second derivative \( f'' \).

Put up the graph of a function \( f, f', \) and \( f'' \). Label the graphs (insert an annotation in the graphics), so that you remember which one is which. Then go illustrate the tests for detecting extrema and inflection points, while enforcing the correct logic.

Use DfW for routine homework: Learn how to fully integrate the utilities of DfW in solving homework problems. Here is an example. Suppose you try to find the geometry of a cylindrical can with maximal volume and a given surface area. Here are the steps:

1. Your Name (annotation)
2. Formulate the problem (annotation).
3. Name the variables (say: \( V \) for volume, \( S \) for surface area, \( r \) for radius, \( h \) for height) (annotation).
4. Relate the variables, \( V = \pi r^2 h \) and \( S = 2\pi rh + 2\pi r^2 \), enter these formulas.
5. Solve the equation for \( S \) in terms of \( h \).
6. Substitute \( h \) in the expression for \( V \).
7. Work out the possible range for \( r \), from 0 to \( R \).
8. State your mathematical problem, maximize \( V \) for \( r \) in the interval \([0, R]\) (annotation).
9. Differentiate \( V \) with respect to \( r \).
10. Use the solve command to find the critical points of $V$.

11. Check that you found a local maximum, check the sign of $V''$.

12. Argue why this is the absolute maximum on the given interval.

13. Find the corresponding $h$.

14. Find the ratio $h/r$.

15. Describe the proportions of the can with maximal volume.

16. Check the answer geometrically, i.e., look at the graph of $V(r)$.

Why do it on the computer. First of all, the student has to do only the thinking, the computer does the calculation. The thinking has to be well organized before the student has anything which to feed to the program. The answers are type, and that is neater than most students handwriting. Remind the students to put in descriptive comments (documentation) so that a reader can follow the process. Before declaring success, all parts of the solution can be moved to where they ought to be, so the student hands in a revised solution, not a first draft.

Once the basic structure is there, students can go in and modify the problem. Ask them to rework the problem if the top of the can is made from a thicker (more expensive) material.

**Numerical Integration** Try your favorite example of a function, say $\sin(x^2)$ on the interval $[0, 1]$. It is easy enough to spell out the left and right endpoint, midpoint, trapezoid, and Simpson’s method. Use an equidistant partitioning of the interval. With a reasonable indexing you have $L_n$, $R_n$, $M_n$, $T_n = (L_n + R_n)/2$ and $S_n = T_n + \frac{2M_n}{3}$. Make a table with rows $[n, L_n, R_n, M_n, T_n, S_n]$ and different values for $n$ in the different rows.