Find the derivative of each of these. (DO NOT SIMPLIFY. Calculators are not allowed. Passing is a score of 6. Passed exams will be given to your instructor. Allowed time: 15 minutes.)

1. \( y = (\sqrt{x})^{2/5} \) 
   
   \[ y' = \frac{2}{5} (\sqrt{x}^2)^{4/5} \cdot 2x \]

2. \( y = x^3 \tan x \) 
   
   \[ y' = 3x^2 \tan x + x^3 \sec^2 x \]

3. \( y = x^{1/2} \cos x \) 
   
   \[ y' = \frac{1}{2} x^{-1/2} \cos x - x^{1/2} \sin x \]

4. \( y = (2y - 7)^{1/3} + 18y^2 - 2y^2 \sqrt{5y^2 + 3.2y + 5.7} \)
   
   \[ = 6y^2 - 4(2y - 7)^3 + 0 + \frac{1}{11} (-1.5y^4 + 3.2y + 5.7)^{10/11} \cdot (-6y^2 - 5.5) \]

5. \( y = (x, y)^{3/4} \) 
   
   \[ y' = -\frac{5}{6} \left( \frac{x}{2} \right)^{-6/5} \cdot \frac{1}{2} \]

6. \( y = \frac{\sqrt{x^2 + 1}}{x + 2} \) 
   
   \[ y' = \frac{(x^2 + 1)(x + 2) - (x^2 + y)}{(x + 2)^2} \]
Calculus I (Math 241) – Test 1

Problem 1. [10 Points] Find the equation of the tangent line to the graph of the function \( f(x) = \sin x \cos x \) at the point \( x = \pi/6 \).

\[
\begin{align*}
\frac{d}{dx}(\pi/6) &= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} \\
\frac{d}{dx}(x) &= \cos^2 x - \sin^2 x \\
\frac{d}{dx}(\pi/6) &= \frac{3}{4} - \frac{1}{4} = \frac{1}{2}
\end{align*}
\]

The tangent line is

\[
L(x) = \frac{1}{2} \left( x - \frac{\pi}{6} \right) + \frac{\sqrt{3}}{4}
\]
Problem 2. [10 Points] Use first principles (the definition of the derivative and the computation of the limit of the appropriate difference quotient) to find the derivative of the function \( f(x) = \sqrt{x + 1} \) at \( x = a \) for any \( a > -1 \).

\[
\frac{d}{dx} f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\sqrt{x+1} - \sqrt{a+1}}{x-a} \cdot \frac{\sqrt{x+1} + \sqrt{a+1}}{\sqrt{x+1} + \sqrt{a+1}}
\]

\[
= \lim_{x \to a} \frac{(x+1) - (a+1)}{(x-a)(\sqrt{x+1} + \sqrt{a+1})}
\]

\[
= \lim_{x \to a} \frac{1}{\sqrt{x+1} + \sqrt{a+1}}
\]

\[
\Rightarrow \frac{d}{dx} f(a) = \frac{1}{2\sqrt{a+1}}
\]
Problem 3. [10 Points] Prove that the equation
\[ x^5 + x^4 - x - 2 = 0 \]
has a solution, and provide an interval of length at most 2 in which one solution will lie. (State which theorem you are applying, and make sure to indicate that all of the assumptions in the theorem are satisfied.)

Set \( f(x) = x^5 + x^4 - x - 2 \)

\[ f(1) = -1 \]
\[ f(2) = 32 + 16 - 2 - 2 > 0 \]

As a polynomial, \( f(x) \) is continuous. Also \( f(1) < 0 \) and \( f(2) > 0 \). According to the intermediate value theorem, there exists some \( c \in (1, 2) \) such that \( f(c) = 0 \). This \( c \) is a solution of the equation in the problem.
Problem 4. [10 Points] Find all asymptotes of the function

\[ g(x) = \frac{x(x^2 + 1)}{x^2 - 1} \]

and provide a rough sketch. It should show at least the intercepts and asymptotes, and where the function positive, resp. negative.

\[ g(x) = \frac{x (x^2 + 1)}{(x+1)(x-1)} = x \frac{(1 + \frac{1}{x^2})}{(1 + \frac{1}{x})(1 - \frac{1}{x})} \]

Vertical asymptotes at \( x = \pm 1 \).

Zero at \( x = 0 \).

Sign change at \( x = 0, \pm 1 \).

\( \lim_{x \to 0} g(x) = -\infty \quad \lim_{x \to \pm 1} g(x) = -\infty \).

There is a slant asymptote with slope 1 as \( x \to \pm \infty \) and \( g \to \infty \) as \( x \to \pm \infty \)

Actually,

\[ g(x) = \frac{x^3 + x}{x^2 - 1} = \frac{x^2 + 2x}{x^2 - 1} = x + \frac{2x}{x^2 - 1} \]

so that the slant asymptote is \( y = x \).
Problem 5. [10 Points] Consider the function arcsin \( x \), the inverse of the sine function.

1. Decide on a domain and range for arcsin \( x \). The answer is not unique.
2. Sketch the graph of arcsin \( x \) using your domain.
3. Find arcsin' \( x \), the derivative of the function, with your domain. Show all work required to derive the solution.

\[
\sin (\arcsin x) = x
\]

\[
\cos (\arcsin x) \cdot \arcsin x = 1
\]

\[
\arcsin' x = \frac{1}{\cos (\arcsin x)}
\]

\[
\arcsin' x = \frac{1}{\sqrt{1-x^2}}
\]
Problem 6. [10 Points] A ladder 12 meters long rests against a vertical wall. If the bottom of the ladder is 1 meter from the wall and it slides away from the wall at a speed of 2 meters per second, how fast is the top of the ladder sliding down?

\[ x^2 + y^2 = 12^2 \]
\[ \frac{dx}{dt} = 2 \text{ m/sec} \]

\[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \]
\[ \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} \]

If \( x = 1 \), then \( y = \sqrt{1143} \)

\[ \frac{dy}{dt} = -\frac{1}{\sqrt{1143}} \cdot 2 \text{ m/sec} \]

At the given time the top slides down at a speed of \( \frac{2}{\sqrt{1143}} \) m/sec.
Problem 7. [10 Points] Below you see the graph of the function \( h(x) \). Fill in the table. The answer might be a number, \( \pm \infty \), ‘yes’ or ‘no’, ‘dne’ (does not exist), or ‘dna’ (does not apply).

| \( a \) | \( \lim_{x \to a^-} h(x) \) | \( \lim_{x \to a^+} h(x) \) | \( \lim_{x \to a} h(x) \) | Continuous?
|--------|-----------------|-----------------|-----------------|-----------------
| 2      | d.n.a           | 2               | d.n.a           | yes            |
| 0      | 0               | 2               | d.n.e           | d.n.a          |
| 0.5    | 2               | 2               | 2               | yes            |
| 1      | 2               | 1               | d.n.e           | no             |
| 2      | 2               | 2               | 2               | no             |
| 3      | 3               | \( \infty \)    | d.n.e           | no             |
| 4      | 1               | 1               | 1               | yes            |