1. Prove that if a module $M$ satisfies the maximum principle then every submodule of $M$ is finitely generated.

2. Let $R$ be the ring $K[X_1, X_2, \ldots]$ of polynomials in the countably many variables $X_i$, $i = 1, 2, 3, \ldots$ Let $I$ be the ideal generated by all the variables $X_i$. (One can also describe $I$ as consisting of all those polynomials with trivial constant term.) Prove that $I$ is not a finitely generated ideal.

3. A family of submodule $\{M_i\}_{i \in I}$ is called directed if for each pair $i, j \in I$ there exists $k \in I$ such that $M_i, M_j \subseteq M_k$. Prove that if $\{M_i\}_{i \in I}$ is a directed family of submodules of $M$ then $\bigcup_I M_i$ is a submodule of $M$.

4. Let $K$ be a field and let $R$ be the ring of two-by-two upper triangular matrices
\[
\begin{pmatrix}
a_{11} & a_{12} \\
0 & a_{22}
\end{pmatrix}
\] with entries in $K$. Write $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

a) Let $M$ be an $R$-module. Prove that $jM = j(e_2M)$ and that $\dim jM \leq \min\{\dim e_1 M, \dim e_2 M\}$. (Here $jM = \{jm | m \in M\}$ and $\dim X$ means the dimension of $X$ considered as a vector space over $K$. If you need to, you can consider only the case where these dimensions are finite.)

b) Prove that if $U$ is a $K$-subspace of $M$ (considering $M$ as a vector space over $K$), then $U$ is an $R$-submodule of $M$ if and only if $e_1 U \subseteq U$, $e_2 U \subseteq U$, and $j U \subseteq U$.

c) Prove that if $r$, $s$, or $t$ are cardinal numbers (or positive integers, if you don't understand cardinals) such that $t \leq \min\{r, s\}$, then there exists an $R$-module $M$ such that $\dim e_1 M = r$, $\dim e_2 M = s$, and $\dim j M = t$.

d) Prove that if $M$ and $N$ are $R$ modules, then $M$ and $N$ are isomorphic if and only if $\dim e_1 M = \dim e_1 N$, $\dim e_2 M = \dim e_2 N$, and $\dim j M = \dim j N$.

e) Consider the three following $R$-modules: $K \oplus 0$, $0 \oplus K$, and $K \oplus K$, where in each case the first summand equals $e_1 M$ and the second equals $e_2 M$; for the first two modules, multiplication by $j$ is defined to be trivial, and in the third case it corresponds to the identity map from $K$ to $K$. (E.g. the $R$-multiplication on $K \oplus K$ is determined by the rules $e_1(x, y) = (x, 0)$, $e_2(x, y) = (0, y)$ and $j(x, y) = (y, 0)$.) Prove that every $R$-module is isomorphic to a direct sum of copies of these three modules.
4. a) Simple computation shows that $j = je_2$. Then $\dim jM \leq \dim e_1M$ because $jM \subseteq e_1M$, and $\dim jM \leq \dim e_2M$ because $jM = j(e_2M)$ and multiplication by $j$ is a linear transformation, so the dimension of the image is at most the dimension of the domain.

b) “⇒” is self-evident, and “⇐” follows from the fact that every $r \in R$ has the form $k_1e_1 + k_2e_2 + k_3j$ with $k_i \in K$.

c) We start with vector spaces $V$ and $W$ with $\dim V = r$, $\dim W = s$. Construct a linear transformation $\theta : W \rightarrow V$ such that $\dim \theta(W) = t$. (The way to construct such an $\theta$ is by considering a basis for $W$. Since $W$ is a free $K$-module, we know from a theorem proved in class that there exists a linear transformation $\theta$ sending the basis elements of $W$ to any elements we choose in $V$.) Now represent elements of $V \oplus W$ by column vectors $\begin{pmatrix} v \\ w \end{pmatrix}$. We can make $V \oplus W$ into an $R$-module by defining the scalar multiplication as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} a_{11}v + a_{12}\theta(w) \\ a_{22}w \end{pmatrix}.$$  

Using the rules for matrix multiplication, one can easily check that the axioms for an $R$-module are satisfied.

d) “⇒” is routine, since if $\varphi$ is an isomorphism from $M$ to $N$, then $\varphi$ restricts to isomorphisms from $e_1M$, $e_2M$, and $jM$ onto $e_1N$, $e_2N$, and $jN$ respectively (why?).

Now suppose the dimensions match. We will construct an isomorphism $\varphi$ from $M$ onto $N$. The difficult part will be to make sure it is $R$-linear.

Choose a basis for $M$ as a vector space in the following way: Choose a basis $m_1, \ldots, m_r \in e_2M$ for the kernel of the linear transformation $e_2M \rightarrow e_1M$ given by $x \mapsto jx$. (The notation suggests that $r$ is finite, but this is not really essential to the proof.) Extend this to a basis for $e_2M$ by choosing additional basis elements $m_{r+1}, \ldots, m_s$. Then $s - r = \dim jM$ and the elements $jm_{r+1}, \ldots, jm_s$ form a linearly independent set in $e_1M$ (why?). In fact, $m_1, \ldots, m_r, m_s, jm_{r+1}, \ldots, jm_s$ form a linearly independent set in $M$ (why?). Finally, choose additional elements $m_{s+1}, \ldots, m_t$ in $e_1M$ so that

$$m_1, \ldots, m_r, m_{r+1}, \ldots, m_s, jm_{r+1}, \ldots, jm_s, m_{s+1}, \ldots, m_t$$

is a basis of $M$ as a $K$-vector space. (Why is this possible?) Then choose a corresponding basis $n_1, \ldots, n_r, \ldots, n_s, jn_{r+1}, \ldots, jn_s, \ldots, n_t$ in exactly the same way for $N$. The reason we can get the subscripts for the $m$’s and $n$’s to match exactly is because of the hypothesis on the various dimensions.
We know that we can define a $K$-linear transformation $\varphi : M \to N$ by specifying where basis elements go. So choose $\varphi(m_i) = n_i$ and $\varphi(jm_i) = jm_i$ for all values of $i$ such that this makes sense. Then $\varphi$ is an isomorphism of vector spaces because it maps one basis to another. The crucial part is to see that $\varphi$ is $R$-linear, and for this it suffices to see that $\varphi(e_i m) = e_i \varphi(m)$ and $\varphi(j m) = j \varphi(m)$ for all $m \in M$. Since $e_i^2 = e_i$, it can easily be seen that $\varphi(e_i m) = e_i \varphi(m)$ for all $m \in M$ if and only if $\varphi(e_i M) \subseteq e_i M$ for $i = 1, 2$. This is clear from our construction of $\varphi$. Furthermore, one easily sees (!) that it suffices to check the equation $\varphi(jm) = j \varphi(m)$ for basis elements. But this is clear from the construction of $\varphi$, if one remembers that $j m_i = 0$ for $i = 1, ..., r$ and $i = s + 1, ..., t$, and likewise for the $n_i$.

e) It is perhaps less confusing to denote the three modules in question by

$\begin{pmatrix} K \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ K \end{pmatrix}$, and $\begin{pmatrix} K \\ K \end{pmatrix}$. For $\begin{pmatrix} K \\ 0 \end{pmatrix}$ and $\begin{pmatrix} K \\ K \end{pmatrix}$, the scalar multiplication is just ordinary matrix multiplication:

$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{22}y \end{pmatrix}$.

For $\begin{pmatrix} 0 \\ K \end{pmatrix}$ the multiplication is given by

$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ a_{22}y \end{pmatrix}$.

(NOTE: The set $L$ of matrices of the form $\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}$ is a left ideal (in fact, two-sided) in $R$. It is easy to see that $\begin{pmatrix} 0 \\ K \end{pmatrix}$ is isomorphic to $R/L$.)

Now let $M$ be an $R$-module and choose a basis for the vector space $M$ in the same form as in part d). For $i = 1, ..., r$, let $M_i$ be the one-dimensional $K$-vector space generated by $m_i$. Since by assumption $m_i \in e_2 M$ and $jm_i = 0$, we easily check that $e_k M_i \subseteq M_i$ and $j M_i \subseteq M_i$, so that $M_i$ is an $R$-submodule of $M$. Furthermore $M_i \approx \begin{pmatrix} 0 \\ K \end{pmatrix}$. Likewise construct $M_i$ for $i = s + 1, ..., t$ as the one-dimensional $K$-space generated by the corresponding $m_i$ and notice that these $M_i$ are also $R$-submodules of $M$ and $M_i \approx \begin{pmatrix} K \\ 0 \end{pmatrix}$. Finally, for $i = r + 1, ..., s$, let $M_i$ be the two dimensional $K$-space generated by $m_i$ and $j m_i$. Then $e_i M_i \subseteq M_i$ and $j M_i = e_1 M_i \subseteq M_i$, so these are also $R$-submodules of $M$ and in this case $M_i \approx \begin{pmatrix} K \\ K \end{pmatrix}$.

It should be clear that $M$ is the direct sum of all these $M_i$. (NOTE: Obviously we could have derived d) from e).