1. **a)** Prove that if \( \{ L_i \}_{i \in I} \) is a family of left ideals in a ring \( R \) such that \( R \) is generated as an \( R \)-module by \( \bigcup I L_i \), then there is a **finite** subset \( J \subseteq I \) such that \( R \) is generated by \( \bigcup J L_i \). (**HINT:** \( R \) is a cyclic \( R \)-module generated by 1.)

**b)** Prove that if a ring \( R \) has a minimal (proper) left ideal \( L \) which is not contained in any proper (two-sided) ideal, then \( R \) is a **finite** direct sum of left ideals isomorphic to \( L \) and \( R \) has no non-trivial proper ideals.

2. **a)** Let \( R_1 \) and \( R_2 \) be ideals in a ring \( R \) such that \( R = R_1 \times R_2 \). (Refer to problem 2 of the October 7 homework and also to pp. 130–131 of Hungerford.) Prove that there exist ideals \( 0 \neq I, J \subseteq R_1 \) such that \( R_1 = I \oplus J \) if and only if there exists an idempotent \( e \) in the center of \( R \) such that \( 0 \neq e \in R_1 \) and \( eR_1 \subset R_1 \).

**b)** Suppose that \( R = R_1 \times \cdots \times R_s = P_1 \times \cdots \times P_t \), where the \( R_i \) and \( P_j \) are indecomposable as (two-sided) ideals (or, equivalently, do not contain any central idempotent except the one that generates them). Prove that \( s = t \) and the \( P_j \) can be renumbered so that for all \( i, R_i = P_i \).

**Definition.** An \( R \)-module \( M \) is called **simple** if \( M \neq 0 \) and \( M \) has no non-trivial proper submodules.

3. Prove that a left \( R \)-module is simple if and only if it is isomorphic to \( R/L \), where \( L \) is a maximal left ideal.

**Fact.** It can be shown using Zorn’s Lemma that if \( r \) is a non-invertible element in a ring then there exists a maximal left ideal containing \( r \) and a maximal right ideal containing \( r \).

4. Let \( R \) be a ring and \( r \in R \). Prove that the following conditions are equivalent and that the set of \( r \in R \) satisfying these conditions forms a (two-sided) ideal.

   (1) \( rM = 0 \) for all simple left \( R \)-modules \( M \).
   (2) \( Nr = 0 \) for all simple right \( R \)-modules \( N \).
   (3) \( \varphi(r) = 0 \) for all homomorphisms \( \varphi \) from \( R \) into a simple left \( R \)-module.
   (4) \( \psi(r) = 0 \) for all homomorphisms \( \psi \) from \( R \) into a simple right \( R \)-module.
   (5) \( r \) belongs to all maximal left ideals in \( R \).
   (6) \( r \) belongs to all maximal right ideals in \( R \).
   (7) For all \( x \in R \), \( 1 - xr \) is invertible.
   (8) For all \( x \in R \), \( 1 - rx \) is invertible.

5. Prove that if a left ideal \( L \) consists entirely of nilpotent elements then \( L \) is contained in every maximal left ideal.

**WARNING:** This does not mean that every nilpotent element in a ring belongs to all maximal left ideals.
1. **b)** If $L$ is not contained in any proper (two-sided) ideal then the ideal generated by $L$ must be $R$. This ideal is $LR$, which is the right ideal spanned by $\bigcup \{ Lr \mid r \in R \}$. By part a), then, there exist elements $r_1, \ldots, r_n \in R$ such that $R = Lr_1 + \cdots + Lr_n$. By omitting any superfluous elements $r_i$ in this sum, we may suppose that for all $i$,

$$Lr_i \not\subseteq Lr_1 + \cdots + \widehat{Lr_i} + \cdots + Lr_n.$$  

Now let $\varphi : L \to Lr_i$ be given by $\ell \mapsto \ell r_i$. Then $\varphi$ is an epimorphism and $\text{Ker} \varphi$ is a left ideal contained in $L$, so since $L$ is a minimal left ideal either $\text{Ker} \varphi = 0$ or $Lr_i \neq 0$ otherwise it could be omitted from the sum above. Thus $\varphi$ is monic, so $Lr_i \approx L$. In particular, $Lr_i$ is a simple left $R$-module. It then follows that

$$Lr_i \cap (Lr_1 + \cdots + \widehat{Lr_i} + \cdots + Lr_n) = 0. \quad (\text{why?})$$

Since this holds for all $i$ it follows that $R = Lr_1 \oplus \cdots \oplus Lr_n$.

Now suppose that $I$ is a non-trivial ideal in $R$. Note that since $R$ is a finite direct sum of artinian modules (the $Lr_i$ are certainly artinian, since they are simple), then $R$ is artinian. It then follows that $I$ must contain a minimal left ideal $L'$. Thus $I$ must contain the ideal $L'R$ generated by $L'$. But as seen above, $L'R = R$. Thus $I = R$. 