1. Let $M$ be a left $R$-module. Show that the following conditions are equivalent:
   (1) Every left invertible $\varphi$ in $\text{End}_R M$ is also right invertible.
   (2) Every right invertible endomorphism is also left invertible.
   (3) No proper summand of $M$ is isomorphic to $M$.

2. a) Prove that a module with finite length over any ring is finitely generated.
    b) Prove that if $R$ is an artinian ring without zero divisors then $R$ is a skew field.
    c) Prove that if $p$ is a prime ideal in a commutative ring $R$ and $R/p$ is an artinian $R$-module then $p$ is maximal.
    d) Prove that if $M$ is a non-trivial module with finite length over a commutative ring $R$, then $\text{Ass} M$ is not empty and consists of maximal ideals.

3. Let $M = K \oplus N$ be a direct sum of $R$-modules.
   a) Prove that if $L$ is a submodule of $M$ such that $N \subseteq L$, then $L = N \oplus (L \cap K)$.
   b) A submodule $H$ of $M$ is called fully invariant if $\varphi(H) \subseteq H$ for all $\varphi \in \text{End}_R M$. Prove that if $H$ is a fully invariant submodule of $M$ then $H = (H \cap K) \oplus (H \cap N)$.
   c) Prove that if $H$ is a direct summand of $M$ (i.e. $M = H \oplus H'$ for some $H'$) and $H$ is fully invariant in $M$, then $H \cap N$ is a direct summand of $N$.

4. Let $L$ be a minimal (non-trivial) left ideal in a ring $R$ and suppose that at least one maximal left ideal of $R$ does not contain $L$.
   a) Prove that $L$ is a direct summand of the left $R$-module $R$.
   b) Prove that $L$ is generated by an idempotent (i.e. by an element $e$ such that $e^2 = e$).

5. Let $L$ be a minimal (non-trivial) left ideal in a ring $R$ and suppose that there exists $x \in L$ such that $Lx \neq 0$. Prove that $L$ is generated by an idempotent.
   (HINT: Prove that the map $\varphi \in \text{End}_R L$ given by $\ell \mapsto \ell x$ is an automorphism of $L$.)