Non-definition. A category is made up of objects and morphisms. For the moment, it won’t hurt to think of an object as being a set together with some sort of structure on that set, and a morphism as consisting of a function between two objects which is compatible with the structure.

Examples of categories. (1) The category of modules over a fixed ring $R$. The objects are $R$-modules and the morphisms are $R$-linear homomorphisms.

(2) The category of rings. The objects are rings and the morphisms are homomorphisms of rings.

(3) The category of not-necessarily-abelian groups.

(4) The category of topological spaces. An object is a set plus a topology on that set. A morphism is a function which is continuous with respect to the topologies.

(5) The category of sets. An object is a set and any function is a morphism.

Some functors. (1) The “forgetful functor” from the category of $R$-modules (for a fixed ring $R$) to the category of abelian groups. If $M$ is an $R$-module then $F(M) = M$, where now we think of $M$ as an abelian group, using the same definition of $m_1 + m_2$ as before. Furthermore, if $\varphi: M \to N$ is an $R$-linear map then $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$, so $\varphi$ is a homomorphism of abelian groups and it makes sense to define $F(\varphi) = \varphi$.

(2) If $R$ is an integral domain and $M$ is an $R$-module, then we can define $F(M)$ to be the torsion submodule of $M$. Now if $\varphi: M \to N$ is $R$-linear then it is easily to see that $\varphi$ maps the torsion submodule of $M$ into the torsion submodule of $N$, so we can define $F(\varphi): F(M) \to F(N)$ to be the “restriction” of $\varphi$. (Note that both the domain and codomain are restricted.)
(3) Let $I$ be a (two-sided) ideal in a ring $R$. If $M$ is an $R$-module then $IM$ is a submodule of $M$. Thus it makes sense to define $F(M) = M/IM$.
Furthermore, if $\varphi : M \to N$ is $R$-linear then $\varphi$ induces a well defined (why?) map $\tilde{\varphi} : M/IM \to N/IN$ given by $\tilde{\varphi}(m + IM) = \varphi(m) + IN$. Thus we can define $F(\varphi) = \tilde{\varphi}$.

Note that $M/IM$ can be thought of as an $R/I$-module by defining a scalar multiplication by $(r + I)(m + IM) = rm + IM$. (CHECK THAT THIS IS WELL DEFINED!)
Furthermore, $F(\varphi)$ as defined above will be $R/I$-linear. Thus we can think of $F$ as a functor from the category of $R$-modules into the category of $R/I$-modules.

If $\varphi$ is a monomorphism [epimorphism] does $F(\varphi)$ have to be monic [epic]?

No.  (1) Let the functor $F$ be defined on the category of $\mathbb{Z}$-modules by letting $F(M)$ be the torsion submodule of $M$. Let $\varphi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the quotient map, where $n > 1$. Then $F(\mathbb{Z}) = 0$ and $F(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. Thus $\varphi$ is an epimorphism but $F(\varphi) : 0 \to \mathbb{Z}/n\mathbb{Z}$ is not.

(2) Let $R$ be a ring and let $I$ be a non-trivial proper ideal in $R$. Let $F(X) = X/IX$ (Example (3) above). Let $N$ be an $R$-module and let $\varphi : IN \to N$ be the inclusion map. Then as described in Example (3) above, $F(\varphi) : IN/I^2N \to N/IN$ will be given by $F(\varphi)(x + I^2N) = x + IN$. But if $x \in IN$ then $x + IN = 0 \in N/IN$. Thus $F(\varphi) = 0$. Hence if $IN \neq I^2N$ (for instance if $I \neq I^2$ and $N = R$) then $F(\varphi)$ is not monic, even though $\varphi$ is monic.

(NOTE: This example illustrates the importance of distinguishing between the inclusion map $\varphi : IN \to N$ and the identity map $1_{IN} : IN \to IN$. $F(1_{IN}) = 1_{IN/I^2N} \neq 0$.)