IN THIS COURSE ALL RINGS AND MODULES ARE UNITARY.
Furthermore, if $R$ and $S$ are rings and $\rho: R \to S$ is a ring morphism, it is required that $\rho(1) = 1$. Likewise if we say that $R$ is a subring of $S$ this means (among other requirements) that they have the same identity element.

1. a) Let $\rho: R \to R'$ be a morphism of commutative rings. Prove that if $p'$ is a prime ideal in $R'$ then $\rho^{-1}(p')$ is prime in $R$.

b) Let $R$ be a subring of $R'$. Prove that if $p'$ is a prime ideal in $R'$ then $p' \cap R$ is a prime ideal in $R$.

Notation. If $p$ is a prime ideal in a commutative ring $R$ and $S = R \setminus p$, then we write $M_p = S^{-1}M$.

Lemma [Hungerford, Theorem 2.2, p 378]. If $S$ is a multiplicative set in a commutative ring $R$ and $a$ is an ideal such that $a \cap S = \emptyset$ then there exists at least one ideal $p$ maximal with respect to the properties $p \supseteq a$ and $p \cap S = \emptyset$. Furthermore, any such ideal is prime.

2. Let $M$ be an $R$-module and $m \in M$ and let $\text{ann} \, m = \{ r \in R \mid rm = 0 \}$. Prove that $m/1 \neq 0 \in S^{-1}M$ if and only if $S \cap \text{ann} \, m = \emptyset$.

3. Let $M, N, P$ be modules over a commutative ring $R$. Prove that:

(1) If $m_1, m_2 \in M$, then $m_1 = m_2$ if and only if $m_1/1 = m_2/1 \in M_m$ for all maximal ideals $m$.

(2) $M = 0$ if and only if $M_m = 0$ for all maximal ideals $m$.

(3) Suppose that $N, P \subseteq M$. Then $N = P$ if and only if $N_m = P_m$ for all maximal ideals $m$.

(4) If $\varphi \in \text{Hom}_R(M, N)$ then $\varphi$ is a monomorphism [epimorphism] if and only if $\varphi_m: M_m \to N_m$ is monic [epic] for all maximal ideals $m$. 
6. Recall the following result [Hungerford, Theorem 2.2, p378]: If \( S \) is a multiplicative set in a commutative ring \( R \) and \( a \) is an ideal such that \( a \cap S = \emptyset \) then there exists at least one ideal \( p \) maximal with respect to the properties \( p \supseteq a \) and \( p \cap S = \emptyset \). Furthermore, any such ideal is prime.

\((\Rightarrow)\): Suppose that \( \text{Ass} M = \{p\} \). Then by the Lemma given in the homework, for every \( m \neq 0 \in M \), \( \text{ann} m \subseteq p \) and every prime ideal containing \( \text{ann} m \) contains \( p \). From this it follows first that if \( s \notin p \) then \( sm \neq 0 \), showing that \( m \notin \ker \theta \), where \( \theta: M \to M_p \) is the canonical map. Furthermore, let \( r \neq 0 \in p \) and let \( S = \{r^k \mid k \geq 1\} \). If \( S \cap \text{ann} m = \emptyset \) then by the above result from Hungerford there exists a prime ideal \( q \) with \( q \supseteq \text{ann} m \) and \( q \cap S = \emptyset \). But then \( q \supseteq p \) and \( r \notin q \), a contradiction. Thus there exists an element \( r^k \in S \cap \text{ann} m \), so \( r^km = 0 \). Since \( p \) is finitely generated (because \( R \) is noetherian) it then follows easily that \( p^{k'}m = 0 \) for some \( k' \).

\((\Leftarrow)\): Now suppose the stated conditions hold and let \( q \in \text{Ass} M \). Then \( q = \text{ann} m \) for some \( m \). By assumption, for some \( k \), \( p^k \subseteq \text{ann} m = q \) Then \( p \subseteq q \) since \( q \) is prime. On the other hand, if \( s \notin p \) then \( sm \neq 0 \) since \( M \to M_p \) is monic and thus \( s \notin q \). Therefore \( q = p \).