1. Let $R$ and $R'$ be rings.
   a) Prove that there exists a multiplication on $R \otimes \mathbb{Z} R'$ such that
      \[(r_1 \otimes r'_1)(r_2 \otimes r'_2) = r_1r_2 \otimes r'_1r'_2.\] Thus $R \otimes \mathbb{Z} R'$ can be considered as a ring.
   b) Prove that if $R$ and $R'$ are commutative, then the diagram
      \[
      \begin{array}{ccc}
      \mathbb{Z} & \longrightarrow & R' \\
      \downarrow & & \downarrow \\
      R & \longrightarrow & R \otimes \mathbb{Z} R'
      \end{array}
      \]
      is a push-out in the category of commutative rings. (Here the maps out of $\mathbb{Z}$ are the only ones possible and the bottom horizontal map, for instance, is given by $r \mapsto r \otimes 1$.)

2. Let $A$ be an $R$-algebra and let $f \in R[X]$ and $B = R[X]/(f)$. By abuse of notation, we will also use $f$ to denote its image in $A[X]$. Prove that $A \otimes_R B \approx A[X]/(f)$.

3. If $C$ is the field of complex numbers, prove that $i \otimes 1 + 1 \otimes i$ is a zero divisor in $C \otimes \mathbb{R} C$.

4. Let $E$ and $F$ be extensions of a field $k$, let $m = [E : k]$ and $n = [F : k]$. (These may possibly be infinite cardinals.) Then we can think of $E \otimes_k F$ as either a $k$-algebra, an $E$-algebra, or an $F$-algebra.
   a) Prove that $\dim_E E \otimes_k F = n$, $\dim_F E \otimes_k F = m$ and $\dim_k E \otimes_k F = mn$.
   b) Prove that if $e_1, \ldots, e_r$ are elements of $E$ linearly independant over $k$, and $f_1, \ldots, f_s$ are $k$-linearly independant elements of $F$, then the $rs$ elements $e_1 \otimes f_1$, $e_1 \otimes f_2, \ldots, e_r \otimes f_s$ of $E \otimes_k F$ are linearly independant over $k$.
   c) Prove that if $F \subseteq F'$ are fields then the inclusion $F \hookrightarrow F'$ induces a monomorphism $E \otimes_k F \hookrightarrow E \otimes_k F'$, so that it is reasonable to think of $E \otimes_k F$ as a $k$-subalgebra of $E \otimes_k F'$.
   d) If $E$ and $F$ are both contained in an extension field $K$ of $k$, prove that $E$ and $F$ are linearly disjoint over $k$ [Lang, §X.5, p. 379] [Hungerford, §VI.2, p. 318] \(\iff E \otimes_k F \text{ is a field, and in this case } E \otimes_k F \approx EF.\)