1. **Two more universal (and couniversal) constructions.**

Consider a square

\[
\begin{array}{ccc}
L & \xrightarrow{\varphi'} & N \\
\downarrow\psi' & & \downarrow\psi \\
M & \xrightarrow{\varphi} & P \\
\end{array}
\]

Define \( \sigma: L \to M \oplus N \) and \( \tau: M \oplus N \to P \) by

\[
\sigma(\ell) = (\psi'(\ell), -\varphi'(\ell)) \quad \text{and} \quad \tau(m, n) = \varphi(m) + \psi(n).
\]

a) Prove that the square commutes if and only if \( \tau \sigma = 0 \).

b) Prove that the following conditions are equivalent:

1. \( 0 \to L \xrightarrow{\sigma} M \oplus N \xrightarrow{\tau} P \) is exact.
2. The square commutes and
   \[
   (\forall m \in M, n \in N) \quad \{ \varphi(m) = \psi(n) \iff (\exists \ell \in L) n = \varphi'(\ell), m = \psi'(\ell) \}.
   \]
3. The square commutes and whenever \( \alpha: X \to M \) and \( \beta: X \to N \) are maps (for any \( R \)-module \( X \)) such that \( \varphi \alpha = \psi \beta \), then there exists a unique map \( \theta: X \to L \) such that \( \alpha = \psi' \theta \) and \( \beta = \varphi' \theta \).

\[
\begin{array}{ccc}
X & & L \\
& \xrightarrow{\psi'} & \xrightarrow{\psi} \\
L & \xrightarrow{\varphi'} & N \\
\downarrow\psi' & & \downarrow\psi \\
M & \xrightarrow{\varphi} & P \\
& & Y
\end{array}
\]

c) Prove that the following conditions are equivalent:

1. \( L \xrightarrow{\sigma} M \oplus N \xrightarrow{\tau} P \to 0 \) is exact.
2. The square commutes, \( P = \varphi(M) + \psi(N) \) and
   \[
   (\forall m \in M, n \in N) \quad \{ \varphi(m) = \psi(n) \iff (\exists \ell \in L) n = \varphi'(\ell), m = \psi'(\ell) \}.
   \]
   (Note that in this case \( \ell \) need not be unique.)
3. The square commutes and whenever \( \gamma: M \to Y \) and \( \delta: N \to Y \) are maps (for any \( R \)-module \( Y \)) such that \( \gamma \psi' = \delta \varphi' \), then there exists a unique map \( \zeta: P \to Y \) such that \( \gamma = \zeta \varphi \) and \( \delta = \zeta \psi \).

**Definition.** If the equivalent conditions in b) are satisfied, we say that the square above is a **pull-back** (Hungerford, p. 484).

If the conditions in c) are satisfied, we say that it is a **push-out**.
2.  a) Prove that if the square in problem 1 is a pull-back, then $\ker \varphi' \cong \ker \varphi$.
   
   b) Prove that if the square in problem 1 is a push-out, then $\coker \varphi' \cong \coker \varphi$.
   
   (NOTE: Coker $\varphi = P/\varphi(M)$.)

   c) Show that Noether’s Second Isomorphism Theorem (Hungerford, Theorem 1.9 (i), p. 173) is a special case of part b).

3. Consider the following commutative diagram with exact rows.

$$
0 \to K \to L \to N \to 0
$$

$$
\begin{array}{ccc}
0 & \to & K \\
\| & & \psi \\
\| & & \| \\
0 & \to & K
\end{array}
\begin{array}{ccc}
\to & M & \to P \\
\psi' & & \psi \\
\to & N & \to 0
\end{array}
$$

Prove that the right hand square is both a pull-back and a push-out.

4. Let $S$ be a multiplicative set in a commutative noetherian ring $R$ and let $M$ be an $R$-module. Prove that

$$
\text{Supp}_{S^{-1}R} S^{-1}M = \{ pS^{-1}R \mid p \in \text{Supp} M \ & \ p \cap S = \emptyset \} 
$$

$$
\text{Ass}_{S^{-1}R} S^{-1}M = \{ pS^{-1}R \mid p \in \text{Ass} M \ & \ p \cap S = \emptyset \} 
$$

$$
\text{Ass}_R S^{-1}M = \{ p \mid p \in \text{Ass} M \ & \ p \cap S = \emptyset \}. 
$$

5. Let $M$ be a module over a commutative noetherian ring $R$ such that $\text{Ass} M$ consists of maximal ideals.

   a) Prove that $\text{Ass} M = \text{Supp} M$.

   b) Prove for every $p \in \text{Ass} M$, the canonical map $M \to M_p$ is a surjection and

   $$
   M_p \cong \{ m \in M \mid (\exists k) \ p^k m = 0 \}. 
   $$

   c) Prove that the family of maps $M \to M_p$ for $p \in \text{Ass} M$ induces an isomorphism

   $$
   M \cong \bigoplus_{\text{Ass} M} M_p. 
   $$
1. Consider a square

\[
\begin{array}{ccc}
L & \xrightarrow{\varphi'} & N \\
\downarrow{\psi'} & & \downarrow{\psi} \\
M & \xrightarrow{\varphi} & P
\end{array}
\]

Define \( \sigma: L \to M \oplus N \) and \( \tau: M \oplus N \to P \) by

\[
\sigma(\ell) = (\psi'(\ell), -\varphi'(\ell)) \quad \text{and} \quad \tau(m, n) = \varphi(m) + \psi(n).
\]

c) \((1)\) \( L \xrightarrow{\sigma} M \oplus N \xrightarrow{\tau} P \to 0 \) is exact.

(3) The square commutes and whenever \( \gamma: M \to Y \) and \( \delta: N \to Y \) are maps (for any \( R \)-module \( Y \)) such that \( \gamma \varphi' = \delta \varphi' \), then there exists a unique map \( \zeta: P \to Y \) such that \( \gamma = \zeta \varphi \) and \( \delta = \zeta \psi \).

**Proof:** (3) \( \Rightarrow \) (1): Proof that \( \text{Ker} \ \tau \subseteq \sigma(L) \):

Note that \( \sigma(L) = \{(\psi'(\ell), -\varphi'(\ell)) \mid \ell \in L\} \).

Consider the following square:

\[
\begin{array}{ccc}
L & \xrightarrow{\varphi'} & N \\
\psi' & & \psi \\
M & \xrightarrow{\varphi} & P
\end{array}
\]

\[
\begin{array}{c}
M \oplus N \\
\xrightarrow{\sigma(L)} \\
\sigma(L)
\end{array}
\]

where \( \gamma(m) = (m, 0) + \sigma(L) \) and \( \delta(n) = (0, n) + \sigma(L) \). Note that

\[
\gamma \psi'(\ell) - \delta \varphi'(\ell) = (\psi'(\ell), -\varphi'(\ell)) + \sigma(L) = 0 \in (M \oplus N)/\sigma(L),
\]

so by hypothesis there exists \( \zeta \) making the diagram commute. Now suppose that \( \tau(m, n) = \varphi(m) + \psi(n) = 0 \). Then

\[
(m, n) + \sigma(L) = \gamma(m) + \delta(n) = \zeta \varphi(m) + \zeta \psi(n) = \zeta(\varphi(m) + \psi(n)) = 0
\]

so \( (m, n) \in \sigma(L) \). \( \square \)
2. b)

\[
\begin{array}{c}
L \xrightarrow{\varphi'} N \xrightarrow{\gamma'} C \xrightarrow{\mu} 0 \\
\psi' \downarrow \quad \psi \downarrow \quad \mu \downarrow \\
M \xrightarrow{\varphi} P \xrightarrow{\xi} D \xrightarrow{0} 0
\end{array}
\]

Let \( C = \text{Coker } \varphi' \) and \( D = \text{Coker } \varphi \). Now \((\xi\psi)\varphi' = \xi\varphi \psi' = 0\) so by the Induced Homomorphism Theorem there exists a unique map \( \mu: C \to D \) making the above diagram commute.

On the other hand, since \( \gamma'\varphi' = 0 = 0\psi' \), by the categorical definition of a push-out there exists a unique map \( \zeta: P \to C \) with \( \zeta\varphi = \gamma' \) and \( \zeta\psi = 0 \). Again by the Induced Homomorphism Theorem \( \zeta \) induces a map \( \eta: D \to C \) such that \( \eta\xi\psi = \zeta\psi = \gamma' \). Then \((\mu\eta\xi)\psi = \mu\zeta\psi = \mu\psi' = \xi\psi \) and \((\mu\eta\xi)\varphi = 0 = \xi\varphi \), so by the definition of a push-out it follows that \( \mu\eta\xi = \xi \), and thus \( \mu\eta = 1_D \) because \( \xi \) is an epimorphism. Also \((\eta\mu)\gamma' = \eta\xi\psi = \zeta\psi = \gamma' \) so \( \eta\mu = 1_C \) because \( \gamma' \) is an epimorphism. Thus \( C \approx D \).

Proof that the square is a push-out: Let \( \gamma: M \to Y \) and \( \delta: N \to Y \) be such that \( \gamma\psi' = \delta\varphi' \). Then \( \gamma\eta = \gamma\psi'\eta' = \delta\varphi'\eta' = 0 \), so by the Induced Homomorphism Theorem there exists a unique \( \zeta: P \to Y \) with \( \zeta\varphi = \gamma \). Furthermore \((\zeta\psi)\varphi' = \zeta\varphi\psi' = \gamma\psi' = \delta\varphi' \). Since \( \varphi' \) is an epimorphism, we conclude that \( \zeta\psi = \delta \). Thus the square in question satisfies the categorical definition of a push-out.

Proof that the square is a pull-back: (Actually, knowing that the square is a push-out, we are already half-way to proving it is a pull-back. But we will start from scratch.) Suppose \( m \in M \) and \( n \in N \) with \( \varphi(m) = \psi(n) \). Since \( \varphi' \) is epic,

\[ (\exists \ell \in L) \quad n = \varphi'(\ell). \]

Then \( \varphi(m - \psi'(\ell)) = \varphi(m) - \varphi\psi'(\ell) = \psi(n) - \psi\varphi'(\ell) = 0 \). Therefore \( m - \psi'(\ell) \in \text{Ker } \varphi \) so by exactness there exists a unique \( k \in K \) with \( m - \psi'(\ell) = \eta(k) = \psi'\eta'(k) \). Thus

\[ m = \psi'(\eta'(k) + \ell) \quad \text{and} \quad n = \varphi'(\eta'(k) + \ell). \]

Furthermore, \( \eta'(k) + \ell \) is the unique element in \( L \) that works. In fact if \( m = \psi'(\ell') \) and \( n = \varphi'(\ell') \), then \( \varphi'(\eta'(k) + \ell - \ell') = 0 \) so by exactness

\[ (\exists k' \in K) \quad \eta'(k) + \ell - \ell' = \eta'(k'). \]
and so \(0 = \psi'(\eta'(k) + \ell - \ell') = \psi'\eta'(k') = \eta(k')\) and so \(k' = 0\) because \(\eta\) is monic, so \(\ell' = \eta'(k) + \ell\).

4. There are three relevant observations:
   (1) The primes of \(S^{-1}R\) are precisely the ideals of the form \(S^{-1}p\), where \(p\) is a prime ideal of \(R\) such that \(p \cap S = \emptyset\). (If \(p \cap S \neq \emptyset\) then \(S^{-1}p = R\).)
   (2) If \(p\) is a prime in \(R\) then \(S^{-1}p \subseteq S^{-1}R\) and in fact \(S^{-1}p = pS^{-1}R\).
   (3) If \(S \cap p = \emptyset\) then \(S \subseteq R \setminus p\) and so \(S^{-1}M_p \approx M_p \approx S^{-1}M_{S^{-1}p}\) (where the LHS can be interpreted in two ways and these two interpretations agree).

   We now see immediately that if \(p \cap S = \emptyset\) then \(S^{-1}M_{S^{-1}p} \neq 0\) and if only if \(M_p \neq 0\).

Another observation: If \(m \in M\), then the annihilator in \(S^{-1}R\) of \(m/1\) is \(S^{-1}(\text{ann } m) \subseteq S^{-1}R\). In fact, if \(r \in \text{ann } m\) then \(rm = 0\) and for any \(s \in S\), \((r/s)(m/1) = rm/s = 0\), so \(r/s \in \text{ann } S^{-1}R(m/1)\). Conversely, if \((r/s)(m/1) = 0\) then \(s'rm = 0\) for some \(s' \in S\). Then \(s'r \in \text{ann } m\) and \(r/s = s'r/s \in S^{-1}(\text{ann } m)\).

Now let \(p \in \text{Ass } M\) and \(p \cap S = \emptyset\). Then \(p = \text{ann } m\) for some \(m \in M\) and so \(S^{-1}p = \text{ann } (m/1) \in \text{Ass } S^{-1}R\).

Conversely, a prime \(\mathfrak{p} = \text{ann } S^{-1}R(m/s)\) in \(\text{Ass } S^{-1}R\) must have the form \(S^{-1}p\), where \(p = \theta_{R}^{-1}(\mathfrak{p})\). Thus for each \(p \in \mathfrak{p}\) there exists \(s' \in S\) with \(s'pm = 0\). Since \(p\) is finitely generated, there exists \(s'' \in S\) with \(s''p m = ps''m = 0\). Thus \(p \subseteq \text{ann } s''m\).

On the other hand, if \(r \in \text{ann } s''m\) then \(rs''m = 0\) and so \(rs''m/s = 0\) so \(r/1 \in S^{-1}p\) and thus \(r \in \theta_{R}^{-1}(S^{-1}p) = p\). Therefore \(p = \text{ann } s''m\).

Now suppose \(p \cap S = \emptyset\) and \(p \in \text{Ass } R\) and \(p = \text{ann } m\). We claim that \(p = \text{ann } (m/1)\).

In fact, for \(r \in R\), \(r \in \text{ann } (m/1)\) if and only if \(srm = 0\) for some \(s \in S\), and this holds if and only if \(sr \in \text{ann } m = p\) for some \(s \in S\), but this is true if and only if \(r \in p\) since \(p\) is prime and \(s \notin p\). Thus \(p \subseteq \text{ann } S^{-1}M\).

Conversely, suppose \(p\) is a prime in \(R\) and \(p \in \text{Ass } S^{-1}M\). Then \(p = \text{ann } (m/1)\) for \(m/1 \in S^{-1}M\). Since \(m/1 \neq 0\) (otherwise \(\text{ann } (m/1) = R\)), it follows that \(sm \neq 0\) for all \(s \in S\) and thus \(p \cap S = \emptyset\). Furthermore for each \(p \in \mathfrak{p}\), \(pm/1 = 0\) so \(spm = 0\) for some \(s \in S\). Since \(p\) is finitely generated it follows that there exists \(s \in S\) such that \(spm = psm = 0\). Therefore \(p \subseteq \text{ann } sm\). On the other hand, clearly \(\text{ann } sm \subseteq \text{ann } (s^m/s) = p\). So \(p = \text{ann } sm \in \text{Ass } M\).  

5. b) To see that \(M \to M_p\) is a surjection it suffices to see that for all prime ideals \(q\), the induced map \(M_q \to (M_p)_q\) is a surjection. This is clear if \(p \not\subseteq q\) since in that case by problem 4, \(p \not\subseteq \text{Ass } M_q = \text{Supp } M_q\) and so \((M_p)_q = (M_q)_p = 0\). Since \(p\) is maximal, this leaves only the case \(q = p\), which is trivial.

**A More Straightforward Proof:** Let \(m/s \in M_p\), where \(s \notin p\). By Problem 3, \(\text{Ass } M_p = \{p\}\), i.e. \(M_p\) is \(p\)-primary, so by a previous homework problem \(p^k(m/1) = 0\)
for some \( k \geq 1 \). Now since \( p \) is maximal, the only prime containing \( p^k \) is \( p \). It follows that \( R/p^k \) is a local ring with maximal ideal \( p/p^k \). Since \( s \notin p \), thus the image of \( s \) in \( R/p^k \) is invertible. Hence there exists \( s' \notin p \) such that \( ss' \equiv 1 \pmod{p^k} \). Then \((ss' - 1)m = 0 \) so that in \( M_p \),

\[
\frac{m}{s} = \frac{s'm}{1} = \theta(s'm),
\]

showing that \( \theta : M \to M_p \) is surjective.

Now let \( M(p) = \{ m \in M \mid (\exists k)p^km = 0 \} \). As seen in the previous paragraph, if \( s \notin p \) then \( s \) is invertible modulo \( p^k \) for all \( k \), so if \( m \in M(p) \) then \( sm \neq 0 \) for all \( s \notin p \) and therefore \( m/1 \neq 0 \in M_p \). This shows that \( \theta : M \to M_p \) restricts to a monomorphism from \( M(p) \) into \( M_p \).

Now let \( m/1 \in M_p \). As previously noted, there exists \( k \geq 1 \) such that \( p^k(m/1) = 0 \). Thus for each \( r \in p^k \), \( rm/1 = 0 \) so there exists \( s \notin S \) such that \( srm = 0 \). Since \( p \) is finitely generated, it follows that there exists \( s \notin p \) with \( sp^km = p^ksm = 0 \), showing that \( sm \in M(p) \). Furthermore, as previously seen, there exists \( s' \notin p \) such that \( ss' \equiv 1 \pmod{p^k} \). Then \( ss'm \in M(p) \) and

\[
\frac{m}{1} = \frac{ss'm}{1}.
\]

Therefore \( \theta(M(p)) = M_p \) and so \( M_p \cong M(p) \).

\( c \) The family of maps \( M \to M_p \) induces

\[
\zeta : M \to \prod_{\text{Ass } M} M_p
\]

(where each coordinate of \( \zeta(m) \) is given by \( m/1 \in M_p \)). Now note that for any \( m \in M \), \( \text{Supp } Rm = \text{Ass } Rm \) is finite, i.e. there are only finitely many prime ideals \( p \) such that \( m/1 \neq 0 \in M_p \). This shows that the image of \( \zeta \) is in fact contained in \( \bigoplus_{\text{Ass } M} M_p \). It now suffices to see that for every prime ideal \( q \), the induced map

\[
M_q \to \bigoplus_{\text{Ass } M} (M_p)_q
\]

is an isomorphism. But as seen in part \( b \), this reduces to the identity map \( M_q \to M_q \).

**A more conventional proof:** It is easy to see that the family of submodules \( M(p) \) of \( M \) is independent, so \( \bigoplus_{\text{Ass } M} M(p) \subseteq M \). Now let \( m \in M \). For each of the finitely many primes \( p_i \), such that \( m/1 \neq 0 \in M_p \), then exists \( k_i \) such that \( p_i^{k_i}m/1 = 0 \in M_{p_i} \).

Then \( p_1^{k_1} \cdots p_n^{k_n}m = 0 \). Now since \( p_i + p_j = R \) for \( i \neq j \), no maximal ideal can contain all the ideals \( a_1, \ldots, a_n \), where

\[
a_i = p_1^{k_1} \cdots \tilde{p}_i^{\tilde{k_i}} \cdots p_n^{k_n},
\]

so \( a_1 + \cdots + a_n = R \) and there exist elements \( a_i \in a_i \) with \( a_1 + \cdots + a_n = 1 \). Furthermore, \( p_i^{k_i}a_im = 0 \) so \( a_iM \in a_iM_p \subseteq M(p_i) \).

Thus \( m = a_1m + \cdots + a_nM \in \bigoplus_{\text{Ass } M} M(p) \), showing that \( M = \bigoplus M(p) \). √