Definition. A family \( \mathcal{F} \) of subsets of a set \( X \) is called a **filter** on \( X \) if it satisfies the following three conditions:

1. \( \emptyset \notin \mathcal{F} \).
2. \( F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F} \).
3. \( F_1 \in \mathcal{F} \) and \( F_1 \subseteq F \subseteq X \Rightarrow F \in \mathcal{F} \).

A filter \( \mathcal{F} \) which is maximal with these properties is called an **ultrafilter**. An ultrafilter \( \mathcal{F} \) is called **principal** if \( \bigcap \mathcal{F} \neq \emptyset \). (In this case, \( \bigcap \mathcal{F} = \{x\} \) for some \( x \in X \) and \( \mathcal{F} = \{F \subseteq X \mid x \in F\} \).)

Lemma. A filter \( \mathcal{F} \) is an ultrafilter if and only if for all \( F \subseteq X \), either \( F \in \mathcal{F} \) or \( X \setminus F \in \mathcal{F} \).

1. Let \( K \) be a field and \( X \) an infinite set.
   a) Complete the proof given in class showing that there is a one-to-one correspondence between the ideals in \( \prod_X K \) and the filters on \( X \), and that prime ideals correspond to ultrafilters.
   b) Characterize those primes corresponding to principal ultrafilters.
   c) Show that all prime ideals in \( \prod_X K \) are maximal.

2. Prove that the following conditions are equivalent for a commutative ring \( R \):
   (1) \( R \) has no non-trivial nilpotent elements and every prime ideal in \( R \) is maximal.
   (2) For every prime ideal \( p \) of \( R \), \( R_p \) is a field.
   (3) For every \( r \in R \) there exists \( x \in R \) with \( r^2x = r \).
   (4) Every finitely generated ideal in \( R \) is generated by an idempotent.

   (Hint: This problem uses several different pieces of the theory we’ve developed.)

3. Prove that if \( p \) is an ideal with height 0 in a commutative (not necessarily noetherian) ring \( R \), then \( p \) consists of zero divisors.

4. Suppose that \( p_0 \) and \( p \) are prime ideals in \( \mathbb{Z}[X] \) such that \( 0 \not\subseteq p_0 \not\subseteq p \).
   a) Identify \( \mathbb{Z} \) as a subring of \( \mathbb{Z}[X] \) in the obvious way. Prove that \( p \cap \mathbb{Z} \neq 0 \).
   b) Prove that there exists a prime number \( p \in \mathbb{Z} \) such that \( p \in p \).
   c) Prove that \( p \) is a maximal ideal.