1. a) Let $A$ be an $R$-algebra and let $G$ be an $R$-module. Recall that we consider $\text{Hom}_R(A, G)$ as an $A$-module, where for $\varphi \in \text{Hom}_R(A, G)$ and $a \in A$ we define $a\varphi$ by $(a\varphi)(x) = \varphi(ax)$. Prove that the isomorphism $\omega_M: \text{Hom}_A(M, \text{Hom}_R(A, G)) \to \text{Hom}_R(M, G)$ defined for $A$-modules $M$ by $\omega_M(\alpha)(m) = \alpha(m)(1)$ is natural in with respect to the variable $M$, i.e. that whenever $\mu: M \to M'$ the following diagram commutes:

\[
\begin{array}{ccc}
\text{Hom}_A(M', \text{Hom}_R(A, G)) & \longrightarrow & \text{Hom}_A(M, \text{Hom}_R(A, G)) \\
\omega_{M'} \downarrow & & \omega_M \\
\text{Hom}_R(M', G) & \xrightarrow{\mu^*} & \text{Hom}_R(M, G)
\end{array}
\]

b) Prove in detail that if $G$ is an injective $R$-module then $\text{Hom}_R(A, G)$ is an injective $A$-module.

c) Let $M$ be an $A$-module, $G$ be an $R$-module, and $\gamma: M \to G$ an $R$-linear mapping. Define $\hat{\gamma}: M \to \text{Hom}_R(A, G)$ by the following composition of maps:

$M \cong \text{Hom}_A(A, M) \hookrightarrow \text{Hom}_R(A, M) \xrightarrow{\gamma} \text{Hom}_R(A, G)$, where the first map is the usual isomorphism and the second is the inclusion map. For $m \in M$ and $a \in A$, what is $\hat{\gamma}(m)(a)$? Prove that if $\gamma$ is a monomorphism, then so is $\hat{\gamma}$.

2. Let $R$ be a commutative ring and $A = R[X]$, the ring of polynomials in one variable with coefficients in $R$.

a) Prove that if $V$ is an $R$-module then $\text{Hom}_R(A, V) \approx \prod_0^\infty V$, where the $A$-module structure is given as follows: if $f \in \prod_0^\infty V$ then $X^n f$ is the sequence $f$ shifted $n$ spaces to the left; i.e. the $k$th coordinate of $X^n f$ is the same as the $k + n$th coordinate of $f$.

b) Prove directly (without using theorems from class) that for any $A$-module $M$, $\text{Hom}_A(M, \prod_0^\infty V) \approx \text{Hom}_R(M, V)$.

c) Prove directly (without using problem 1) that if $R$ is a field, then $\text{Hom}_R(A, V)$ is always injective.

d) Let $M$ be an $A$-module, let $\varphi \in \text{Hom}_R(A, M)$, and let $f \in \prod_0^\infty M$ correspond to $\varphi$. Prove that $\varphi$ is $A$-linear $\iff (\forall n) f_n = X^n f_0$.

e) Let $M$ be an $A$-module and define $\rho: M \to \prod_0^\infty M$ as the composition $M \cong \text{Hom}_A(A, M) \hookrightarrow \text{Hom}_R(A, M) \xrightarrow{\sim} \prod_0^\infty M$, where the first map is the usual
isomorphism. Prove that $\prod_{0}^{\infty} M$ is not (in most cases) an essential extension of $\rho(M)$.

f) Assuming that $R$ is a field compute the injective envelope of $\rho(M)$ in $\prod_{0}^{\infty} M$.

**The Symmetric Algebra.** If $R$ is a commutative ring and $M$ is an $R$-module, define $T_R(M) = \bigoplus_{n=0}^{\infty} M^\otimes n$, where $M^\otimes 0 = R$ and for $n > 1$, $M^\otimes n = M \otimes_R \cdots \otimes_R M$ ($n$ factors). With multiplication defined in the obvious way, this is called the **tensor algebra** for $M$ over $R$. Now let $I$ be the ideal generated by all elements of the form $m \otimes m' - m' \otimes m$ for $m \in M$.

(Not that, for instance,

$$m_1 \otimes m_2 \otimes m_3 \equiv m_2 \otimes m_1 \otimes m_3 \equiv m_2 \otimes m_3 \otimes m_1 \equiv m_3 \otimes m_2 \otimes m_1 \pmod{I}.$$)

The $R$-algebra $S_R(M) = T_R(M)/I$ is commutative and is called the **symmetric algebra** of $M$ over $R$.

3. Prove that if $F$ is a free $R$-module of rank $n$ ($n < \infty$) then $S_R(F) \approx R[X_1, \ldots, X_n]$.

4. Prove that $S_R(M)$ is characterized by the following universal property:

There is a homomorphism of $R$-modules $\delta: M \to S_R(M)$ and if $A$ is any commutative $R$-algebra and $\varphi: M \to A$ is a homomorphism of $R$-modules, then there is a unique homomorphism $\mu$ of $R$-algebras such that the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\delta} & S_R(M) \\
\downarrow & & \downarrow \mu \\
S_R(M) & \rightarrow & A.
\end{array}
$$