We will show how to derive formulas $Q = \int_a^b g(x) \, dx$, where $Q$ is a given variable depending on a continuous function $g(x)$ defined between $x = a$ and $x = b$. We assume that $Q$ is increasing with respect to the function $g$, i.e. making $g$ bigger always makes $Q$ bigger. We will also assume that $Q = 0$ whenever $a = b$.

Basically what will be seen is that if a formula of this sort gives the right answer for a particular variable $Q$ whenever $g(x)$ is a constant function, then it will work for all continuous functions $g(x)$.

Hopefully, the examples that follow will make clear what is meant by a variable $Q$ which depends on a function $g$.

Examples.

1. $Q$ is the area under the graph $y = g(x)$ between the endpoints $a$ and $b$.

2. $Q$ is the distance traveled between time $a$ and time $b$ by an object moving at (variable) speed $g(x)$.

3. $Q$ is the volume of the solid obtained by revolving around the $x$-axis the graph of the function $f(x)$ between the endpoints $a$ and $b$. In this case, we will choose $g(x) = \pi f(x)^2$.

4. $Q$ is the volume of the solid obtained by revolving around the $y$-axis the graph of the function $f(x)$ between the endpoints $a$ and $b$. In this case, we will choose $g(x) = 2\pi xf(x)$.

5. $Q$ is the work done by a force $F = g(x)$ acting on an object moving between points $x = a$ and $x = b$. 
Solids of Revolution. The canonical application of integration is the problem of finding the volume of the solid obtained by revolving the area under the graph of a function \( y = g(x) \) around the \( x \)-axis or \( y \)-axis.

Our point of view is that a solid of revolution is simply a misshapen cylinder. It is well known that the volume of a cylinder is given by the formula \( V = \pi R^2 H \), where \( R \) is the radius and \( H \) the height.

If the cylinder is positioned horizontally, so that the \( x \)-axis becomes its axis, with the base at position \( x = a \) and the other end at \( x = b \), then this formula can be written as

\[
V = \pi R^2 H = \pi R^2 (b - a) = \pi \int_a^b R^2 \, dx.
\]

Now if we now allow the radius \( R \) to become a variable quantity \( g(x) \) as the variable \( x \) moves through the cylinder, then we now have a solid of revolution and the formula for the volume should be modified to read

\[
V = \pi \int_a^b g(x)^2 \, dx.
\]

If a cylinder with height \( H \) is positioned vertically, so that the \( y \)-axis becomes its axis, and if the radius is given by \( R = b \), then the volume can be given by the formula

\[
V = \pi R^2 H = \pi b^2 H = \pi \int_0^{b^2} H \, d(x^2) = 2\pi \int_0^b xH \, dx.
\]

(The first integral looks slightly strange, but we can think of it as simply a shorthand for a change of variables \( u = x^2 \). In other words, for practical purposes, \( d(x^2) = 2x \, dx \).) If we now allow the height \( H \) to become a variable quantity \( h(x) \) as the variable \( x \) moves from the center of the cylinder outwards, then the formula for volume should be modified to read

\[
V = \pi \int_0^{b^2} h(x) \, d(x^2) = 2\pi \int_0^b xh(x) \, dx.
\]

There is a charm to this way of deriving the standard formulas for the volume of a solid of revolution, but at first the thinking behind it seems a little dubious. However it can in fact be justified.
**THEOREM.** Suppose that $Q$ is a quantity that depends on a function $g(x)$ defined between $a$ and $b$. (In formal notation, we can write $Q = Q(g, a, b)$.) Suppose further that $Q$ is increasing as a function of $g$ (i.e. making the function $g$ larger always makes $Q$ larger) and is additive over disjoint intervals, and that whenever $g(x) = m$, where $m$ is a constant, then for all values $a$ and $b$, $Q = (b - a)m$. Then it will be true that for every specific choice of a continuous function $g$,

$$Q = \int_a^b g(x) \, dx.$$  

**Proof:** Consider any given function $g(x)$. Hold $a$ fixed and write $Q(x)$ for the value $Q$ takes when we consider the function between $a$ and $x$ instead of between $a$ and $b$. Since $Q(a) = 0$, by the Fundamental Theorem of Calculus

$$Q(b) = Q(b) - Q(a) = \int_a^b Q'(x) \, dx.$$  

Therefore it suffices to prove that $Q'(x) = g(x)$. Now

$$Q'(x) = \lim_{h \to 0} \frac{Q(x + h) - Q(x)}{h}.$$  

If $g$ were a constant function $g(x) = m$, then the basic assumption about $Q$ could be applied to the interval with endpoints $x$ and $x + h$ [or the interval from $x - h$ to $x$ in case $h$ is negative] to show that $Q(x + h) - Q(x) = mh$ in this special case. For a non-constant function $g(x)$, if $m$ is the minimum value that $g$ takes between $x$ and $x + h$ and $M$ is the maximum, then $m \leq g(x') \leq M$ for all $x'$ between $x$ and $x + h$ and so applying the constant function case to the constants $m$ and $M$ yields $mh \leq Q(x + h) - Q(x) \leq Mh$ (because of the assumption that $Q$ increases when the function gets larger), and so

$$m \leq \frac{Q(x + h) - Q(x)}{h} \leq M.$$  

Now $m$ and $M$ actually depend on $h$, and since $g$ is continuous they both converge to $g(x)$ when $h$ approaches 0 (with $x$ being held constant):

$$\lim_{h \to 0} m(h) = \lim_{h \to 0} M(h) = g(x).$$  

Thus by the Pinching Theorem,

$$g(x) = \lim_{h \to 0} m(h) \leq \frac{Q(x + h) - Q(x)}{h} = Q'(x) \leq \lim_{h \to 0} M(h) = g(x).$$  

Therefore $Q'(x) = g(x)$ and so $Q(b) = \int_a^b Q'(x) \, dx = \int_a^b g(x) \, dx$.  

\[ \Box \]
Application to examples. (1) Since the area under the graph of a horizontal line $g(x) = m$ between $x = a$ and $x = b$ is just $(b - a)m$, the theorem shows that the area under the graph of any function $g(x)$ between the endpoints $x = a$ and $x = b$ is $\int_{a}^{b} g(x) \, dx$.

(2) Since the distance traveled between time $x = a$ and $x = b$ by an object moving at a constant speed $m$ is $(b - a)m$, the theorem shows that the distance traveled between time $x = a$ and time $x = b$ by an object moving at a variable speed $g(x)$ is $\int_{a}^{b} g(x) \, dx$.

(3) If graph of a constant function $f(x) = m$ between $x = a$ and $x = b$ is revolved around the $x$-axis, the solid obtained is a horizontal cylinder with radius $m$ and length $b - a$, so the volume is $\pi (b - a)m^2$. Thus the theorem shows that the volume of the solid obtained by revolving around the $x$-axis the graph of the function $f(x)$ between the endpoints $a$ and $b$ is $\pi \int_{a}^{b} f(x)^2 \, dx$.

(4) If the graph of a constant function $f(x) = m$ between $x = a$ and $x = b$ is revolved around the $y$-axis, the solid obtained is a vertical cylindrical shell with inner radius $a$ and outer radius $b$. Its volume is $(\pi b^2 - \pi a^2)m$, which can be also written as $2\pi \int_{a}^{b} mx \, dx$. Although this doesn’t quite fit the pattern of the theorem, the same logic shows that the volume of the solid obtained by revolving around the $y$-axis the graph of any function $f(x)$ between the endpoints $a$ and $b$ is $2\pi \int_{a}^{b} xf(x) \, dx$.

(5) The work done by a constant force $g(x) = m$ acting on an object moving between $x = a$ and $x = b$ is $(b - a)m$. Thus the theorem shows that the work done by a variable force $g(x)$ acting on an object moving from $x = a$ to $x = b$ is $\int_{a}^{b} g(x) \, dx$. 