MAXIMAL MATCHINGS

**Given** a set of \( n \) men and \( m \) women such that certain pairs of a man and women are acceptable partners. We say that such a pair is **compatible**.

Now, without assuming the Marriage Condition, we will determine the maximal number of compatible pairs that can be matched.

For each man \( x_i \), let \( A_i \) be the set of women who are compatible with \( x_i \).

Choose a maximal matching \( M \), i.e. no possible matching contains more compatible pairs than \( M \). Let \( \rho \) be the number of pairs matched under \( M \).

**Lemma.** For every set \( i_1, \ldots, i_k \) of integers between 1 and \( n \),

\[
\rho \leq |A_{i_1} \cup \cdots \cup A_{i_k}| + n - k.
\]

**Proof:** There can be at most \( |A_{i_1} \cup \cdots \cup A_{i_k}| \) matchings in \( M \) involving the men \( x_{i_1}, \ldots, x_{i_k} \), since that’s all the women available for them. On the other hand, there can be at most \( n - k \) additional matchings, since that’s all the men that are left. Since there are \( \rho \) men matched under \( M \), it follows that \( \rho \leq |A_{i_1} \cup \cdots \cup A_{i_k}| + n - k \). \( \square \)

Use the procedure in the proof of the Marriage Theorem to assign a star to either the man or the woman in each matched couple in such a way that every compatible pair (whether matched or not) contains at least one starred person.

**Lemma.** Let \( \{x_{i_1}, \ldots, x_{i_k}\} \) be the set of unstarred men. Then \( \rho = |A_{i_1} \cup \cdots \cup A_{i_k}| + n - k \).

**Proof:** Since \( \{x_{i_1}, \ldots, x_{i_k}\} \) are unstarred, all the compatible partners for \( \{x_{i_1}, \ldots, x_{i_k}\} \) must be starred, i.e. all the women in \( A_{i_1} \cup \cdots \cup A_{i_k} \) must be starred (otherwise there would be a compatible pair with neither partner starred). Furthermore, there are \( n - k \) starred men. But there is one star for each matched pair, so \( \rho \) is the total number of stars. Thus \( \rho \geq |A_{i_1} \cup \cdots \cup A_{i_k}| + n - k \). By the preceding lemma, we know that \( \rho \leq |A_{i_1} \cup \cdots \cup A_{i_k}| + n - k \). Thus \( \rho = |A_{i_1} \cup \cdots \cup A_{i_k}| + n - k \). \( \square \)

To summarize, we have
**Theorem.** The maximum possible number of men that can be matched under any matching is the smallest value taken by $|A_{i_1} \cup \cdots \cup A_{i_k}| + n - k$ over all sets $\{i_1, \ldots, i_k\}$ of integers between 1 and $n$.

**Proof:** The maximum number of men that can be matched under any matching is $\rho$. We have seen that for all sets $\{i_1, \ldots, i_k\}$, $\rho \leq |A_{i_1} \cup \cdots \cup A_{i_k}| + n - k$ and that there exists at least one such set (corresponding to the set of unstarred men in the preceding lemma) such that $\rho = |A_{i_1} \cup \cdots \cup A_{i_k}| + n - k$. $\Box$