PLEASE RETURN TEST SHEET

Do any seven problems.

1. Prove that a positive integer $n$ is prime if and only if $\varphi(n) = n - 1$.

2. Prove that there do not exist integers $x$ and $y$ such that $x^2 + y^2 \equiv 3 \pmod{4}$.

3. Prove that a positive integer $m$ is even if and only if $\varphi(2m) = 2\varphi(m)$.

4. Find the continued fraction expansions for the following real numbers:
   a) $\sqrt{20}$
   b) $\frac{42}{29}$

5. Prove that if $n$ is any natural number and $s$ is the sum of the digits of $n$, then $n$ and $s$ give the same remainder when divided by 9.
   (In other words, if $a_k, \ldots, a_0$ are the digits when $n$ is written in its normal decimal representation and if $n = 9q + r$ with $0 \leq r < 9$, then $a_k + \cdots + a_0 = 9q_2 + r$ for some $q_2$.)

6. Let $a$ be an integer and $p$ a prime. Prove that if $x^2 - x + a$ is not a multiple of $p$ for all integers $x$ with $0 \leq x < p$ then $x^2 - x + a$ is not a multiple of $p$ for any $x$ whatsoever.
7. We know that if \( d = (a, b) \) then there exist integers \( x \) and \( y \) such that \( ax + by = d \). Use this to prove that the linear Diophantine equation \( ax + by = c \) has a solution if and only if \( (a, b) \mid c \).

8. Prove that the simple continued fraction expansion for \( \sqrt{a^2 - 1} \) (where \( a \geq 2 \)) is \([a - 1; 1, 2a - 2] \). 

9. Let \( p \) be a prime number of the form \( 4k + 1 \) and let \( q \) be any prime number except 2. Prove that \( p \mid x^2 - q \) for some \( x \) if and only if \( q \mid y^2 - p \) for some \( y \).

10. a) Prove that for any \( x \), every prime factor of \( x^2 + 1 \) (except 2) has the form \( 4k + 1 \). In particular, if \( p_1, \ldots, p_n \) are prime numbers, then the odd prime factors of \( p_1^2 \cdots p_n^2 + 1 \) must have the form \( 4k + 1 \).

b) Use this to prove that there are infinitely many primes of the form \( 4k + 1 \).