Recursive Algorithms Illustrating Proofs by Induction

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**Theorem.** A set consisting of \( n \) elements has exactly \( 2^n \) subsets.

The proof of this theorem by induction is illustrated by the following recursive algorithm for listing all the subsets of a set.

**procedure** list-subsets(\( A_n = \{x_1, \ldots, x_n\} \): set with exactly \( n \) elements)

\[
\text{if } n = 0 \text{ then list } \emptyset \\
\text{else begin} \\
\quad \text{list-subsets}(A_{n-1}) \\
\quad \text{Adjoin } x_n \text{ to each subset listed in the line above and list the resulting sets.} \\
\text{end}
\]

\{ All the subsets of \( A_n \) have been listed. \\
The subsets of \( A_{n-1} \) are the same as the subsets of \( A_n \) not containing \( x_n \). \\
Each subset of \( A_{n-1} \) produces exactly one subset of \( A_n \) containing \( x_n \). \\
Thus \( A_n \) has twice as many subsets as \( A_{n-1} \). \}

**Theorem.** A \( 2^n \times 2^n \) checkerboard with one square deleted can be tiled by an L-shaped tile three squares large.

The proof of this theorem by induction is illustrated by the following recursive algorithm for tiling the checkerboard.

**procedure** tile(\( C_n \): a \( 2^n \times 2^n \) checkerboard with one square deleted)

\[
\text{if } n = 1 \text{ then lay down a single tile in the only possible way} \\
\text{else begin} \\
\quad \text{Cut the board up into four } 2^{n-1} \times 2^{n-1} \text{ sub-boards} \\
\quad \text{tile(the sub-board with the missing piece)} \\
\quad \text{for } i = 1 \text{ to } 3 \\
\quad \quad \text{begin} \\
\quad \quad \quad B := \text{the next sub-board not already tiled} \\
\quad \quad \quad B := B \text{ with the central corner removed} \\
\quad \quad \quad \text{tile}(B) \\
\quad \quad \text{end} \\
\quad \text{tile}(the \text{three squares in the middle that are still untiled}) \\
\text{end}
\]