The Column Space & Column Rank of a Matrix

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Let \( A = \begin{bmatrix} 0 & 1 & 2 & -3 & 4 \\ 0 & 2 & 4 & 0 & 20 \\ 0 & -1 & -2 & 5 & 0 \end{bmatrix} \).

The column space of \( A \) is the subspace of \( \mathbb{R}^3 \) spanned by the columns of \( A \), in other words it consists of all linear combinations of the columns of \( A \):

\[
\begin{align*}
&u \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + y \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix} + z \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix},
\end{align*}
\]

It can then be seen that a vector \( b \) with coordinates \( b_1 \), \( b_2 \) and \( b_3 \) belongs to the column space of \( A \) precisely when the system of equations

\[
\begin{bmatrix}
0 & 1 & 2 & -3 & 4 \\
0 & 2 & 4 & 0 & 20 \\
0 & -1 & -2 & 5 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]

has a solution.

Now if we take the augmented matrix for this linear system and reduce it to row echelon form then we get

\[
\begin{bmatrix}
0 & 1 & 2 & -3 & 4 & b'_1 \\
0 & 0 & 0 & 1 & 2 & b'_2 \\
0 & 0 & 0 & 0 & 0 & b'_3
\end{bmatrix}
\]

(where \( b'_1 \), \( b'_2 \), and \( b'_3 \) are new constants).

For instance, if we are trying to solve

\[
\begin{bmatrix}
0 & 1 & 2 & -3 & 4 \\
0 & 2 & 4 & 0 & 20 \\
0 & -1 & -2 & 5 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
3 \\
-6 \\
-7
\end{bmatrix}
\]
then the augmented matrix reduces to the row echelon form
\[
\begin{bmatrix}
0 & 1 & 2 & -3 & 4 & | & 3 \\
0 & 0 & 0 & 1 & 2 & | & -2 \\
0 & 0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
\]
and so the solutions have the form
\[
\begin{bmatrix}
u \\
w \\
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ -10 \\ -2 \\ 0 \\ 1 \end{bmatrix}
\]
where \( r, s, \) and \( t \) are arbitrary constants.

**Important:** For instance, we can get one solution (a particular solution) by setting \( r = s = t = 0 \). In general, **if a solution exists at all, then there is a solution in which all variables are zero which correspond to columns which do NOT contain leading entries in the row echelon form.** (Be sure that you understand why this is true!)

By choosing the solution with \( r = s = t = 0 \) we get
\[
\begin{bmatrix} 3 \\ -6 \\ -7 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} - 2 \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}.
\]

Applying this in general, we can see that **if** \( \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \) is in the column space of \( A \), then it is possible to write
\[
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + y \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}
\]
for the right choice of \( w \) and \( y \), so that \( \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \) can be written as a linear combination of the two columns \( \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \) and \( \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix} \).
Restated, this says that

The column space of $A$ is spanned by the two columns

$$
\begin{bmatrix}
1 \\
2 \\
-1 \\
\end{bmatrix}
\text{ and }
\begin{bmatrix}
-3 \\
0 \\
5 \\
\end{bmatrix}.
$$

Since it can be easily (!) seen that these two vectors are linearly independent, it follows that they are a basis for the column space of $A$. Therefore the column rank of $A$, which by definition is the dimension of the column space, is 2.

As a general principle, let $A$ be any $m \times n$ matrix and let $A_1, \ldots, A_n$ be the columns of $A$, so that $A = [A_1 \ A_2 \ \ldots \ A_n]$. Then these $n$ columns $A_i$ are vectors in $\mathbb{R}^m$ and the column space of $A$ consists of all vectors $b \in \mathbb{R}^m$ which can be written in the form

$$
b = x_1A_1 + \cdots + x_nA_n = A \begin{bmatrix} x_1 \\
\vdots \\
x_n \end{bmatrix}.
$$

Now the procedure for solving systems of linear equations by elimination shows that if it is possible to solve the system

$$
A \begin{bmatrix} x_1 \\
\vdots \\
x_n \end{bmatrix} = b
$$

at all, then there is one solution where only those variables $x_{j_1}, \ldots, x_{j_r}$ corresponding to columns containing the leading entry for some row in the row echelon form of $A$ need to be non-zero. (So here $r$ will be the number of leading entries in the row echelon form, which is the same as the number of non-zero rows in the row echelon form.) This says that if $b$ is in the column space of $A$ then there exist values $x_{j_1}, \ldots, x_{j_r}$ such that $b = x_{j_1}A_{j_1} + \cdots + x_{j_r}A_{j_r}$.

Restated, this says that the column space of $A$ is spanned by the columns $A_{j_1}, \ldots, A_{j_r}$, where $j_1, \ldots, j_r$ are the numbers for the columns which contain the leading entry for some row in the row echelon form of $A$. (Thus $r$ is the number of non-zero rows in the row echelon form of $A$.)

Furthermore, these columns $A_{j_1}, \ldots, A_{j_r}$ will always be linearly independent. In fact, we can see that $x_{j_1}A_{j_1} + \cdots + x_{j_r}A_{j_r} = 0$ is only possible when $x_{j_1} = \cdots = x_{j_r} = 0$ by noticing that the row echelon form of $[A_{j_1} \ \ldots \ A_{j_r}]$ is an $m \times r$ matrix whose row echelon form has a leading entry in each column.

(Why? Hint: How does the row echelon form for $[A_{j_1} \ \ldots \ A_{j_r}]$ compare to the row echelon form of $A = [A_1 \ \ldots \ A_n]$?)