Let $v_1, \ldots, v_p, w$ be vectors in $\mathbb{R}^m$. Let $A$ be the $m \times p$ matrix whose columns are $v_1, \ldots, v_p$:

$$A = [v_1 \ldots v_p]$$

- $w$ belongs to the subspace spanned by $v_1, \ldots, v_p$.
  MEANS
- $w$ is a linear combination of $v_1, \ldots, v_p$.
  MEANS
- There exist scalars $x_1, \ldots, x_p$ such that $w = x_1 v_1 + \cdots + x_p v_p$.
  MEANS
- There exists a vector $\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$ in $\mathbb{R}^p$ such that $A \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = w$.
  MEANS
- The linear system $Ax = w$ has a solution.
  MEANS
- The linear system $Ax = w$ is consistent.

- $v_1, \ldots, v_p$ are linearly independent.
  MEANS
- If $x_1 v_1 + \cdots + x_p v_p = 0$ then $x_1 = 0, \ldots, x_p = 0$.
  MEANS
- If $A \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ then $\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = 0$.
  MEANS
- The homogeneous system $Ax = 0$ has only the trivial solution.
Any linear transformation $L: \mathbb{R}^m \to \mathbb{R}^n$ is given by a formula of the form $L(x) = Ax$, where $A$ is an $m \times n$ matrix. More specifically, the columns of $A$ are $L(v_1), \ldots, L(v_n)$ where $v_i$ is the column vector with 1 in the $i$th coordinate and 0 in the other coordinates.

If $L$ and $A$ are related in this way, then

\[ \ker L \]

IS THE SAME AS

The null space of $A$

IS THE SAME AS

The solution space to the homogeneous system $Ax = 0$.

- $w \in \text{Range } L$
  MEANS

- $w = L(x)$ for some $x \in \mathbb{R}^n$
  MEANS

- The linear system $Ax = w$ has a solution
  MEANS

- $w$ belongs to the column space of $A$.

- $L$ is one-to-one.
  MEANS

- If $L(v) = L(v')$ then $v = v'$.
  MEANS

- For each $w \in \mathbb{R}^m$ there is at most one $x \in \mathbb{R}^n$ such that $L(x) = w$.
  MEANS

- If $w \in \mathbb{R}^m$ then the linear system $Ax = w$ has either a unique solution or no solution at all.