PROPERTIES OF \( \mathbb{R} \)

I now list some properties of \( \mathbb{R} \) implied by our construction.

0.1. **Order relation on \( \mathbb{R} \).** Let \( \{b_n\}_\sim \) and \( \{c_n\}_\sim \) be two real numbers (which we understand as equivalence classes of Cauchy sequences of rational numbers). We define the relation \( \leq \) by

\[
[b_n]_\sim \leq [c_n]_\sim \quad \text{if and only if} \quad ([b_n] \sim [c_n]) \lor (\exists N : (n > N) \Rightarrow (b_n < c_n))
\]

**Exercise** Prove that the relation \( \leq \) on \( \mathbb{R} = \mathbb{C}/\sim \) is well-defined.

**Exercise** Prove that the relation \( \leq \) on \( \mathbb{R} = \mathbb{C}/\sim \) is indeed an order relation, and extends the relation \( \leq \) on \( \mathbb{Q} \subseteq \mathbb{R} \).

0.2. **Operations on \( \mathbb{R} \).** Let \( \{b_n\}_\sim \) and \( \{c_n\}_\sim \) be two real numbers (which we understand as equivalence classes of Cauchy sequences of rational numbers).

We define the operations of addition and multiplication by

\[
\{b_n\}_\sim + \{c_n\}_\sim = \{b_n + c_n\}_\sim
\]

and

\[
\{b_n\}_\sim \cdot \{c_n\}_\sim = \{b_n \cdot c_n\}_\sim
\]

**Exercise** Prove that the sequences on the right are indeed Cauchy sequences (otherwise these definitions make no sense).

**Exercise** Prove that the operations of addition and multiplication are well-defined by the above formulas, and extend the corresponding operations on \( \mathbb{Q} \subseteq \mathbb{R} \).

**Exercise** Think about how to define the operations of subtraction and division. While everything goes pretty smoothly with subtraction, there are substantial difficulties with division. Try to describe and overcome these difficulties.

0.3. **Cauchy sequences of real numbers.** This definition is quite similar to that of Cauchy sequences of rationals.

**Definition.** A sequence \( \{C_n\} \) of real numbers is a Cauchy sequence if

\[
\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : (n > N \land m > N) \Rightarrow |C_n - C_m| < \varepsilon
\]

In this definition, we may still assume that \( \varepsilon \in \mathbb{Q} \). We again define an equivalence relation \( \sim \) on the set \( \mathbb{C}_\mathbb{R} \) of all Cauchy sequences of real numbers in exactly same way as we did that above for rationals.

**Definition.** We say that a Cauchy sequence \( \{C_n\} \) converges to \( \alpha \in \mathbb{R} \) if

\[
\{C_n\} \sim \{\alpha, \alpha, \alpha, \ldots\}
\]

**Exercise** Compare this definition with the definition of a limit of a sequence from Calculus.

**Theorem.** Every Cauchy sequence of real numbers converges to a real number.
Proof. Let \( \{C_n\} \) be a Cauchy sequence of real numbers. That is, every element of the sequence \( C_n \) is an equivalence class of Cauchy sequences of rational numbers. Pick a representative \( \{c_m\} \) from this class (i.e. \( C_n = [\{c_m\}] \sim \)), and consider the sequence \( \{c_n\} \) of rational numbers.

**Exercise** Prove that \( \{c_n\} \) is a Cauchy sequence.

Let \( \alpha = [\{c_n\}] \sim \).

**Exercise** Prove that \( \{C_n\} \sim \{\alpha, \alpha, \alpha, \ldots\} \) as required. □

What we just did in the proof may be interpreted as presenting of a function \( C_\mathbb{R}/\sim \rightarrow C/\sim \).

**Exercise** Present this function explicitly and prove that this is an isomorphism.

### 0.4. Bounded sets

Since we have order relations on our number sets, we can now introduce the notion of bounded sets.

A set \( S \) of numbers is bounded above if there exists a number \( M \) such that \( x \leq M \) for every \( x \in S \).

Similarly, one can introduce the notion of being bounded below.

A number \( u \) is called a sharp upper bound for \( S \) if

\[
(\forall x \in S, \ x \leq u) \land (\forall t < x, \ \exists v \in S : \ t > v).
\]

We already had an example of a subset \( S \subseteq \mathbb{Q} \) which is bounded above and does not have a sharp upper bound in \( \mathbb{Q} \):

\[
S = \{x \in \mathbb{Q} : \ x^2 < 2\}.
\]

Next theorem tells us that this never happens in \( \mathbb{R} \).

**Theorem.** Every non-empty bounded above subset \( S \) of \( \mathbb{R} \) has a sharp upper bound in \( \mathbb{R} \).

**Proof.** Let \( M_0 \) be an upper bound. We will construct the sharp upper bound as a Cauchy sequence of \( \{c_n\} \) elements of our set \( S \). Let \( c_0 \in S \) be an arbitrary element of \( S \). Clearly, \( c_0 < M_0 \). Let

\[
t = \frac{c_0 + M_0}{2}.
\]

Either \( t \) is also an upper bound for \( S \), or there is an element \( c \in S \) such that \( c > t \).

In the first case, we let \( M_1 = t \) and \( c_1 = c_0 \), while in the second case we let \( c_1 = t \) and \( M_1 = M_0 \). Now we produce a new \( t \) with

\[
t = \frac{c_1 + M_1}{2}
\]

and continue recursively producing a new \( t \), new \( c_n \), and new \( M_n \) on each step.

**Exercise** Prove that \( \{c_n\} \) produced in this way is a Cauchy sequence.

**Exercise** Prove that \( \{c_n\} \sim \in \mathbb{R} \) is a sharp upper bound for \( S \).

**Exercise** Prove that a sequence of real numbers \( \{b_n\} \) which is bounded

\[
\exists M : \ \forall n, \ b(n) < M
\]

and increasing

\[
\forall n, \ b(n + 1) > b(n)
\]
is a Cauchy sequence.

0.5. A Cauchy sequence for $\sqrt{2}$. In order to present $\sqrt{2}$ as a real number, we want to construct a Cauchy sequence whose equivalence class is $\sqrt{2}$.

I will give here a way which was discovered by Babylonians several thousand years ago.

The rough idea is as follows. As you know, $\sqrt{2}$ is not a rational number. That is the equation

$$p^2 = 2q^2$$

has no solutions such that both $n$ and $m$ are integers. Let us modify the equation slightly. The equation

$$p^2 = 2q^2 + 1$$

already has a solution in integers, namely $(p, q) = (3, 2)$. If we can find a solution with big enough $p$ and $q$ so that our $^n+1$-modification looks neglectably small, then their ratio, $p/q$ will be pretty close to $\sqrt{2}$.

In order to implement this idea, we will make use of the following statement.

**Proposition.** Let $p, q$ be positive integers such that

$$p^2 = 2q^2 + 1$$

Then

$$(p^2 + 2q^2)^2 = 2(2pq)^2 + 1$$

We now construct a sequence $\{b_n\}$ with the following inductive procedure.

$$p_0 = 3, \quad q_0 = 2,$$

and, for any natural $n$,

$$p_{n+1} = p_n^2 + 2q_n^2, \quad q_{n+1} = 2p_nq_n$$

We let

$$b_n = \frac{p_n}{q_n}.$$ 

**Exercise** Prove that $\{b_n\}$ is a Cauchy sequence.

**Exercise** Consider the real number $x = \lfloor \{b_n\} \rfloor$. Prove that

$$x^2 = 2.$$