THE SET OF FUNCTIONS FROM $A$ TO $B$

Definition 10.9 makes introduces notations which are very natural, and mnemonically acceptable. Indeed, for two finite sets $A$ and $B$, which consist of $|A|$ and $|B|$ elements respectively, their product

$$A \times B$$

has exactly $|A| \times |B|$ elements, while the set of functions from $A$ to $B$

$$B^A$$

has exactly $|B|^{|A|}$ elements.

However, the words preceding Definition 10.9 rather justify the first (quite obvious), not the second (less obvious) of above formulas. It may be difficult to figure out why the second formula is correct. I give here a proof which uses a combination of notions we have learned so far in this class. I thus want to prove the following proposition.

**Proposition.** Let $A$ and $B$ be finite non-empty sets which consist of $|A|$ and $|B|$ elements respectively. The set of functions from $A$ to $B$ consists of $|B|^{|A|}$ elements.

**Proof.** We argue by induction in $|A|$.

We begin with the base case when $|A| = 1$. Clearly, there are exactly $|B| = |B|^{|A|}$ functions as required. Indeed, if $a$ is the unique $a \in A$, then graphs of every function must be a one element subset ${\{(a, b) \in A \times B \text{ with all possible } b \in B}}$, and there are exactly $|B|$ such pairs. (Please pause and think what does that mean in terms of possible images of $a$.)

Now let us pass to induction step. namely we want to prove that for $|A| = n + 1$ $B^A = |B|^{n+1}$

assuming that $B^S = |B|^n$

for any set $S$ out of $n$ elements.

Let $x \in A$ be an element of $A$, and let $S = A \setminus \{x\}$. Then $S \subseteq A$ and $|S| = n$. We introduce a relation $\sim$ on the set of functions $B^A$ by

$$f \sim g \text{ if and only if } \forall s \in S, f(s) = g(s)$$

**Exercise.** Prove that $\sim$ is an equivalence relation on $B^A$.

Since $\sim$ is an equivalence relation, we can consider the set $B^A / \sim$ of equivalence classes. It now suffices to prove the following claims.

**Claim 1.** There are $|B|$ elements in every equivalence class.

**Claim 2.** There are $|B|^{|n| = |B|^{|S|}}$ equivalence classes.

Indeed, if we have these claims then we can calculate the amount of elements in $B^A$ as

$$|B^A| = |B| \times |B|^n = |B|^{n+1} = |B|^{|A|}$$

as required.

Let us now support our claims. Claim 2 follows from the inductive assumption since there are as many equivalence classes as functions from $S$ to $B$. 

\[1\]
Exercise. Explain why this is true.
For Claim 1, note that an equivalence class \([g]_\sim\) is the set of functions

\[\{ f \in B^A \mid \forall s \in S, \ f(s) = g(s) \}\].

For a function \(f\) in this set, its value \(f(x) \in B\) may be any element of \(B\), and there are exactly \(|B|\) of them. We thus have exactly \(|B|\) elements (functions) in an equivalence class. 

\(\square\)