### An Example With Unequal Mixed Partial Derivatives

This example is suggested by Salas and Hille in their textbook, *Calculus, 7th edition*, as problem 43 on page 941:

\[
f(x, y) = \begin{cases} 
xy(y^2 - x^2) / (x^2 + y^2), & \text{for } (x, y) \neq (0, 0) \\
0, & \text{for } (x, y) = (0, 0).
\end{cases}
\]

We shall show that \(f_{x,y}(0,0) = +1\) and \(f_{y,x}(0,0) = -1\).

**Here are the first derivatives:**

1. For \((x, y) \neq (0, 0)\), we can use the quotient rule and simplify to obtain
   \[
f_x(x, y) = \frac{-x^4y - 4x^2y^3 + y^5}{(x^2 + y^2)^2}
   \]

2. For \((x, y) = (0, 0)\), we need to use the basic definition of derivative as a limit to get
   \[
f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{-0^4h - 4(0^2)h^3 + 0^5}{h} = 0
   \]

3. For \((x, y) \neq (0, 0)\), we can use the quotient rule and simplify to obtain
   \[
f_y(x, y) = \frac{-x^5 + 4x^3y^2 + xy^4}{(x^2 + y^2)^2}
   \]

4. For \((x, y) = (0, 0)\), we need to use the basic definition of derivative as a limit to get
   \[
f_y(0, 0) = \lim_{h \to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{-h^5 + 4h^30^2 + 0^4}{h} = 0
   \]

The computation of \(f_{x,y}(0,0)\) is possible by using the limit definition of derivative:

\[
f_{x,y}(0,0) = \lim_{h \to 0} \frac{f_x(0, 0 + h) - f_x(0, 0)}{h}
= \lim_{h \to 0} \frac{-0^4h - 4(0^2)h^3 + 0^5}{(h^2 + 0^2)^2} - 0
= \lim_{h \to 0} \frac{h^5}{h^4} = 1
\]

Here is the computation of \(f_{y,x}(0,0)\):

\[
f_{y,x}(0,0) = \lim_{h \to 0} \frac{f_y(0 + h, 0) - f_y(0, 0)}{h}
= \lim_{h \to 0} \frac{-h^5 + 4h^30^2 + h0^4}{(h^2 + 0^2)^2} - 0
= \lim_{h \to 0} \frac{-h^5}{h^4} = -1
\]