Taylor’s Theorem in Two Variables

**Theorem.** Let \( f \) be in \( C^{n+1}([a, b] \times [c, d]) \). If \((x, y)\) and \((x + h, y + k)\) are points in the rectangle \([a, b] \times [c, d] \subset \mathbb{R}^2\), then

\[
(1) \quad f(x + h, y + k) = \sum_{i=0}^{n} \frac{1}{i!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(x, y) + E_n(h, k)
\]

where

\[
E_n(h, k) = \frac{1}{(n+1)!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(x + \theta h, y + \theta k)
\]

in which \( \theta \) lies in \((0, 1)\).

**Remark 1.** Stating the theorem for a rectangle \([a, b] \times [c, d]\) generalizes to any convex set: basically, the theorem needs a straight line segment inside the domain connecting \((x, y)\) to \((x + h, y + k)\). Moreover, we can relax the continuity of derivatives condition of \( f \) somewhat at the endpoints of such line segments.

**Remark 2.** The phrase \( f \) in \( C^{n+1} \) means that \( f \) has at least \( n+1 \) derivatives (of a types of mixtures) and that every one of them is continuous. By a generalization of Clairaut’s Theorem, order of differentiation won’t matter. For example, when \( n + 1 \geq 4 \),

\[
\frac{\partial^4 f}{\partial x^2 \partial y^2}
\]

is the same whether we do \( f_{x,x,y,y}, f_{x,y,x,y}, f_{y,x,x,y}, f_{y,y,x,x}, f_{y,y,x,x} \) or \( f_{x,y,y,x} \). (The industrious reader can check that there are six different ways to differentiate with respect to \( x \) twice and \( y \) twice.)

**Remark 3.** Here

\[
\left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i = \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^j f}{\partial x^j \partial y^{i-j}}(x, y)h^j k^{i-j}
\]

is a wonderful, shorthand way to write down every possible \( i \)-th derivative with every possible choice of differentiating with respect to \( x \) \( j \) times and with respect to \( y \) \( i-j \) but with (the needed) binomial coefficient

\[
\binom{i}{j} = \frac{i!}{j!(i-j)!}
\]

Note that, when \( i = 0 \), we get just \( f(x, y) \).

**Lemma 1.** Assume that \( f \) is in \( C^{n+1}([a, b] \times [c, d]) \) with \((x, y)\) and \((x + h, y + k)\) in \([a, b]\). Set

\[
g(t) = f(x + th, y + tk), \quad \text{for } t \in [0, 1].
\]

Then

\[
(2) \quad g^{(i)}(t) = \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(x + th, y + tk), \quad \text{for } 0 \leq i \leq n + 1.
\]
Proof. When $i = 0$,

$$g^0(t) = g(t) = f(x + th, y + tk) = \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^0 f(x + th, y + tk)$$

Thus the formula (2) holds for $i = 0$. Suppose that holds for some $i < n + 1$. Based on that, we shall now prove that formula (2) also holds for $i + 1$. We will need a famous binomial identity, called Pascal’s triangle:

$$\binom{i+1}{j} = \binom{i}{j} + \binom{i}{j-1}, \quad \text{for } 0 \leq j \leq i + 1.$$

We shall also use the chain rule on terms of the following form:

$$\frac{d}{dt} \left[ \frac{\partial^i f}{\partial x^j \partial y^{i-j}}(x + th, y + tk) \right] = \frac{\partial}{\partial x} \left( \frac{\partial^i f}{\partial x^j \partial y^{i-j}}(x + th, y + tk) \right) \cdot \frac{d(x + th)}{dt} + \frac{\partial}{\partial y} \left( \frac{\partial^i f}{\partial x^j \partial y^{i-j}}(x + th, y + tk) \right) \cdot \frac{d(y + tk)}{dt}$$

$$= \frac{\partial^{i+1} f}{\partial x^{i+1} \partial y^{i-j}}(x + th, y + tk) \cdot h$$

$$+ \frac{\partial^{i+1} f}{\partial x^{i+1} \partial y^{i-j+1}}(x + th, y + tk) \cdot k$$

Now for the truth about $i + 1$:

$$g^{(i+1)}(t) = \frac{d}{dt} \left[ g^{(i)}(t) \right]$$

$$= \frac{d}{dt} \left[ \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(x + th, y + tk) \right]$$

(here we have used the truth known for $g^{(i)}$)

$$= \frac{d}{dt} \left[ \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^j f}{\partial x^j \partial y^{i-j}}(x + th, y + tk) h^j k^{i-j} \right]$$

(here we have used Remark 3 above)

$$= \sum_{j=0}^{i} \binom{i}{j} \left( \frac{\partial^{i+1} f}{\partial x^{i+1} \partial y^{i-j}}(x + th, y + tk) \cdot h \right) h^j k^{i-j}$$

$$+ \sum_{j=0}^{i} \binom{i}{j} \left( \frac{\partial^{i+1} f}{\partial x^{i+1} \partial y^{i-j+1}}(x + th, y + tk) \cdot k \right) h^j k^{i-j}$$

(here we have used the chain rule as described earlier)

$$= \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^{i+1} f}{\partial x^{i+1} \partial y^{i-j}}(x + th, y + tk) h^{j+1} k^{i-j}$$

$$+ \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^{i+1} f}{\partial x^{i+1} \partial y^{i-j+1}}(x + th, y + tk) h^j k^{i-j+1}$$

(here we grouped factors of $h$ and $k$ for shortened expressions)
Let the first sum on the right side of equation (4) be $A$ and the second sum be $B$. First, we simplify $B$:

$$B = \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^{i+1} f}{\partial x^{j} \partial y^{i-j+1}} (x + th, y + tk)h^{j}k^{i-j+1}$$

$$= \sum_{j=1}^{i} \binom{i}{j} \frac{\partial^{i+1} f}{\partial x^{j} \partial y^{i-j+1}} (x + th, y + tk)h^{j}k^{i-j+1} + \binom{i}{0} \frac{\partial^{i+1} f}{\partial x^{0} \partial y^{i+0-0}} (x + th, y + tk)h^{0}k^{i-0+1}$$

(we set aside the $j = 0$ term)

$$= \sum_{j=1}^{i} \binom{i}{j} \frac{\partial^{i+1} f}{\partial x^{j} \partial y^{i-j+1}} (x + th, y + tk)h^{j}k^{i-j}$$

$$+ \binom{i+1}{0} \frac{\partial^{i+1} f}{\partial x^{0} \partial y^{i-0+1}} (x + th, y + tk)h^{0}k^{i+1-0}$$

In our last steps above, we replaced $i-j+1$ with $i+1-j$ and replaced $\binom{i}{0}$ with $\binom{i+1}{0}$, which is allowed since both equal 1.

Next we simplify $A$:

$$A = \binom{i}{i} \frac{\partial^{i+1} f}{\partial x^{i+1} \partial y^{i-i}} (x + th, y + tk)h^{i+1}k^{i-i}$$

$$+ \sum_{j=0}^{i-1} \binom{i}{j} \frac{\partial^{i+1} f}{\partial x^{j+1} \partial y^{i-j}} (x + th, y + tk)h^{j+1}k^{i-j}$$

(we set aside $j = i$ for special consideration)

$$= \binom{i+1}{i+1} \frac{\partial^{i+1} f}{\partial x^{i+1-0} \partial y^{0}} (x + th, y + tk)h^{i+1-0}k^{0}$$

$$+ \sum_{s=1}^{i} \binom{i}{s-1} \frac{\partial^{i+1} f}{\partial x^{(s-1)+1} \partial y^{(i-(s-1))}} (x + th, y + tk)h^{(s-1)+1}k^{i-(s-1)}$$

$$= \binom{i+1}{i+1} \frac{\partial^{i+1} f}{\partial x^{i+1-0} \partial y^{0}} (x + th, y + tk)h^{i+1-0}k^{0}$$

$$+ \sum_{s=1}^{i} \binom{i}{s-1} \frac{\partial^{i+1} f}{\partial x^{s} \partial y^{i+1-s}} (x + th, y + tk)h^{s}k^{i+1-s}$$

In our last three steps above, we replaced $\binom{i}{i}$ with $\binom{i+1}{i+1}$, which is OK because both equal 1. We also changed $i-i$ to 0 and $i+1$ to $i+1-0$ in the first term. Finally, in the summation we changed the index variable from $j$ to $s = j + 1$; thus $j = s - 1$ is substituted throughout and, because $j$ ranged from 0 to $i-1$, the variable $s$ ranges from 1 to $i$. Finally, we simplified the algebra by replacing $i-(s-1)$ with $i+1-s$ and $(s-1)+1$ with $s$.

We’ve arranged that $A$ and $B$ look more alike. In the summations, $s$ ranges from 1 through $i$ for $A$ and $j$ ranges from 1 to $i$ for $B$. Also, the $j$-th term of $B$ and $s$-th
term of $A$ are almost identical when $j = s$ except that they have slightly different binomial coefficients: $\binom{s - 1}{j}$ for $A$ and $\binom{s}{j}$ for $B$. By Pascal’s triangle, these two coefficients add to $\binom{i + 1}{j}$ when $s = j$. Thus

$$g^{(i+1)}(t) = A + B$$

$$= \binom{i + 1}{i + 1} \frac{\partial^{i+1} f}{\partial x^{i+1} \partial y^s} (x + th, y + tk) h^{i+1-0} k^0$$

$$+ \sum_{j=1}^{i} \binom{i + 1}{j} \frac{\partial^{i+1} f}{\partial x^j \partial y^{i+1-j}} (x + th, y + tk) h^j k^{i+1-j}$$

$$+ \binom{i + 1}{0} \frac{\partial^{i+1} f}{\partial x^0 \partial y^{i+1}} (x + th, y + tk) h^0 k^{i+1-0}$$

$$= \sum_{j=0}^{i+1} \binom{i + 1}{j} \frac{\partial^{i+1} f}{\partial x^j \partial y^{i+1-j}} (x + th, y + tk) h^j k^{i+1-j}$$

$$= \left( \frac{\partial}{h} \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{i+1} f(x + th, y + tk)$$

That completes the proof that the truth for any $i$ in $[0, n + 1)$ gives the truth for $i + 1$. □

Proof of the Theorem. By the one-dimensional Taylor’s Theorem, for each $t$ in $[0, 1]$, there is some $\theta_t$ in $(0, 1)$ such that

$$g(t) = \sum_{i=0}^{n} \frac{g^{(i)}(0)}{i!} t^i + \frac{f^{(n+1)}(\theta_t)}{(n + 1)!} t^{n+1}$$

Let $t = 1$, write $\theta_1$ as simply $\theta$, and use Lemma 1 to express $g^{(i)}(0)$:

$$f(x + h, y + k) = g(1)$$

$$= \sum_{i=0}^{n} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(x + 0 \cdot h, y + 0 \cdot k) \cdot \frac{1^i}{i!}$$

$$+ \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(x + \theta \cdot h, y + \theta \cdot k) \cdot \frac{1^{n+1}}{(n + 1)!}$$

$$= \sum_{i=0}^{n} \frac{1}{i!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(x, y)$$

$$+ \frac{1}{(n + 1)!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(x + \theta \cdot h, y + \theta \cdot k)$$

That completes the proof. □