Some Ramsey Theory in Nonstandard Analysis

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Outline of talk:

1. How does an ‘applied’ nonstandard analyst get sucked into the world of infinitary combinatorics?

2. Reflections on the use of NSA in infinitary combinatorics. (In particular: sketches of nonst’d proofs of results of Rowbottom et al.)
Motivation and History

- Common equation:

\[ \text{Combinatorics} + \text{NSA} = (\text{standard}) \text{ mathematical results} \]
- usually finitary combinatorics

- Some appearances of infinitary combinatorics within NSA:

(I) Advantages to building the nonstandard model using special ultrafilters:

- (Cutland, Kessler, Kopp, R. 1988) TFAE:
  1. \( \forall x \in \mathbb{R}^N/D, \text{ if } x \approx 0 \text{ then } x = ^*s_M \text{ for some standard sequence } s_n \text{ converging to 0 and some infinite } M; \)
  2. \( D \) is a P-point
• (II) Infinitary combinatorics + NSA = (standard) mathematical results:


  – (Benedikt) Recent work on database query languages (W/Keisler, Libkin, et al.) (Ramsey theory, Vapnik-Chervonenkis Dimension)

• (III) Use of NSA to prove Ramsey-like results:

  – (Keisler, Kunen, Miller, Leth 1989) Let \( \Omega \) internal; then any \( F : \mathcal{P}_n(\Omega) \to \mathbb{N} \) with a countably determined graph has an infinite internal homogeneous subset.

Nonst’d proofs of results of Rowbottom et al.

Fix $\kappa$ a cardinal number $\geq \omega$

**Theorem.** If $\kappa$ is measurable then $(\kappa)^n_m$. In other words, if $F : \mathcal{P}_n(\kappa) \rightarrow m$, then there is an $X \subseteq \kappa$ such that $\text{card}(X) = \kappa$ and $X$ is homogenous for $F$. ($n \in \mathbb{Z}^+, m < \kappa$). Moreover, if $\mu$ is normal on $\kappa$ then we may take $\mu(X) = 1$.

**Definition.** A measure $\mu$ on $\kappa$ is *normal* provided that for any $f \in {}^\kappa\kappa$, if $\mu\{\alpha < \kappa \mid f(\alpha) < \alpha\} = 1$ then for some $\alpha_0 \in \kappa$, $\mu(f^{-1}\{\alpha_0\}) = 1$.

**Theorem.** If $\kappa$ is an uncountable measurable cardinal then $\kappa$ is measurable with respect to a normal measure.

**Corollary.** (Erdös-Hajnal) Assume $\kappa$ measurable, $\delta < \kappa$, $\forall \alpha < \delta, n_\alpha \in \mathbb{N}^+$, $m_\alpha < \kappa$, and $F_\alpha : \mathcal{P}_{n_\alpha}(\kappa) \rightarrow m_\alpha$. Then there is an $X \subseteq \kappa$ such that $\text{card}(X) = \kappa$ and $X$ is homogenous for $F_\alpha$ (same $X$ works for all $\alpha$).
If $H \in ^*\kappa$, define $\mu_H : \mathcal{P}(\kappa) \rightarrow 2$ by:

$$
\mu_H A = \begin{cases} 
1, & H \in ^*A \\
0, & \text{otherwise},
\end{cases}
$$

and say that $H$ represents $\mu_H$

**Proposition.** (Nonst’d representation of measures.) If $\mu : \mathcal{P}(\kappa) \rightarrow 2$ is a finitely additive measure then for some $H \in ^*\kappa$, $\mu = \mu_H$.

**Proposition.** (Nonst’d representation of normality.) The following are equivalent:

1. $\mu_H$ is normal
2. for any $f \in ^\kappa\kappa$, if $^*f(H) < H$ then $^*f(H) \in \kappa$.

(In this case, call $H$ normal.)

**Proposition.** (Nonst’d representation of $\alpha$-additivity) The following are equivalent:

1. $\mu_H$ $\alpha$–additive
2. $\forall \{A_i\}_{i<\alpha} \subseteq \mathcal{P}(\kappa), \ H \in (\bigcap_{i<\alpha}^* A_i) \Rightarrow H \in ^*(\bigcap_{i<\alpha} A_i)$.

**Corollary.** If $\mu_H$ is $\alpha$-additive and $f \in ^\kappa\kappa$, then $\mu_{f(H)}$ is $\alpha$-additive.
Theorem. If $H \in {}^{*}\kappa \setminus \kappa$ and $\mu_H$ is $\alpha$–additive, $\alpha \geq \omega$, then for some normal $H_0 \in {}^{*}\kappa \setminus \kappa$, $\mu_{H_0}$ is $\alpha$–additive.

Proof sketch:

Corollary. If $\kappa$ is an uncountable measurable cardinal then $\kappa$ is measurable with respect to a normal measure.
Proof of Rowbottom result:

(Remark. It suffices to assume \( m \) finite, \( n = 2 \)).

Let \( \mu = \mu_H \) be a measure on \( \kappa \). Put \( C_j = \{ x \in \kappa \mid *F(x, H) = j \} \), \( j < m \). Since \( \kappa = C_0 \cup \cdots \cup C_{m-1}, H \in *C_j \) for some \( j \).

Define \( \bigcup_{\alpha} \subseteq \kappa \) inductively, \( \alpha < \kappa \), so that:

(a) \( H \in *\bigcup_{\alpha} \)
(b) \( \{ x_{\alpha} \}_{\alpha < \kappa} \) is increasing, where \( x_{\alpha} = \inf \bigcup_{\alpha} \)
(c) \( F(x_{\beta}, x) = j \) whenever \( x \in \bigcup_{\alpha}, \alpha > \beta \)

Here’s how:

1. \( \bigcup_0 := C_j \).
2. \( \alpha \) a limit: \( \bigcup_{\alpha} := \bigcap_{\beta < \alpha} \bigcup_{\beta} \).
3. \( \alpha = \beta + 1 \): \( \bigcup_{\alpha} := \{ x \in \bigcup_{\beta} \setminus \{ x_{\beta} \} \mid F(x_{\beta}, x) = j \} \)
   (Note \( x_{\beta} \in \bigcup_{\beta} \subseteq C_j \), so \( *F(x_{\beta}, H) = j \), so \( H \in *\bigcup_{\alpha} \).

Put \( X = \{ x_{\alpha} \}_{\alpha < \kappa} \). (c) guarantees that \( F(x, y) = j \) whenever \( x \neq y \in X \), i.e. \( X \) is homogeneous for \( F \).

Suppose now that \( \mu \) is normal but \( \mu X = 0 \). Note that for any \( x \in \bigcup_0 \) there is a greatest \( \beta = \beta(x) < \kappa \) with \( x \in \bigcup_{\beta} \).
(Otherwise, take \( \alpha \) least with \( x \not\in \bigcup_{\alpha} \), then \( \alpha \) is a limit and \( x \in \bigcap_{\beta < \alpha} \bigcup_{\beta} \), a contradiction.) Put \( \varphi(x) = x_{\beta(x)} \), note \( \varphi(x) < x \) for \( x \in \bigcup_0 \setminus X \), so \( *\varphi(H) = \alpha_0 = x_{\beta_0} \) for some \( \alpha_0, \beta_0 \in \kappa \); then \( H \not\in *\bigcup_{\beta_0 + 1} \), a contradiction.
Question: Suppose $f : (\Omega, \mathcal{A}_L, P_L) \rightarrow Y$ is Loeb measurable, $Y$ a metric space; does $f$ have a lifting?

Suffices: (*) If $\mathcal{E}$ partitions $\Omega$ into Loeb nullsets, then for some $\mathcal{E}' \subseteq \mathcal{E}$, $\bigcup \mathcal{E}'$ is not Loeb measurable.

Remark: Follows easily if Loeb measure is compact; this can depend on underlying set theory (Jin, Shelah).

Theorem. (R., 1996) Suppose (i) $\kappa = \text{card}(\mathcal{A})$; (ii) For some nondecreasing sequence $\{\mathcal{A}_i\}_{i<\kappa}$ with each $\mathcal{A}_i \subseteq \mathcal{A}$ compact, $\mathcal{A} = \bigcup_{i<\kappa} \mathcal{A}_i$; and (iii) No $\alpha < \kappa$ is both measurable and cofinal in $\kappa$. Then (*) holds.

Proof start:

Else let $\mathcal{E} = \{E_i\}_{i<\alpha}$ be a counterexample with $\alpha$ least.

Induces a ($\sigma$-additive) measure on $(\alpha, \mathcal{P}(\alpha))$.

Ulam dichotomy:

(1) Atomless: RVMC, diagonalize using a Bernstein set.

(2) An atom: then WOLG $\alpha$ is a measurable cardinal, use Erdős-Hajnal in a clever way.

Question: Can (i)–(iii) in the theorem statement be replaced with (eg) the Isomorphism Property (Henson, Jin, Shelah) or some version of the Generic Filter Property (Di Nasso, Hrbacek)?