SECTION 1

1. Verify each of the following limits.

(i) \( \lim_{n \to \infty} \frac{n}{n + 1} = 1. \)

(ii) \( \lim_{n \to \infty} \frac{n + 3}{n^2 + 4} = 0. \)

(iii) \( \lim_{n \to \infty} \sqrt{n^2 + 1} - \sqrt{n + 1} = 0. \) Hint: You should at least be able to prove that \( \lim_{n \to \infty} \sqrt{n^2 + 1} - \sqrt{n^2} = 0. \)

(iv) \( \lim_{n \to \infty} \frac{n!}{n^n} = 0. \) Hint: \( n! = n(n - 1) \cdots k! \) for \( k < n, \) in particular, for \( k < n/2. \)

(v) \( \lim_{n \to \infty} \sqrt[n]{a} = 1, \quad a > 0. \)

(vi) \( \lim_{n \to \infty} \sqrt[n]{n} = 1. \)

(vii) \( \lim_{n \to \infty} \sqrt[n]{n^2 + n} = 1. \)

(viii) \( \lim_{n \to \infty} \sqrt[n]{a^n + b^n} = \max(a, b). \)

(ix) \( \lim_{n \to \infty} \frac{\alpha(n)}{n} = 0, \) where \( \alpha(n) \) is the number of primes which divide \( n. \) Hint: The fact that each prime is \( \geq 2 \) gives a very simple estimate of how small \( \alpha(n) \) must be.

2. Find the following limits.

(i) \( \lim_{n \to \infty} \frac{n}{n + 1} - \frac{n + 1}{n}. \)

(ii) \( \lim_{n \to \infty} n - \sqrt{n + a\sqrt{n} + b}. \)

(iii) \( \lim_{n \to \infty} \frac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}}. \)
3. (a) What can be said about the sequence \( \{a_n\} \) if it converges and each \( a_n \) is an integer?
(b) Find all convergent subsequences of the sequence 1, \(-1, 1, -1, 1, -1, \ldots\). (There are infinitely many, although there are only two limits which such subsequences can have.)

5. (a) Prove that if \( 0 < a < 2 \), then \( a < \sqrt{2a} < 2 \).
(b) Prove that the sequence
\[
\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots
\]
converges.
(c) Find the limit. Hint: Notice that if \( \lim_{n \to \infty} a_n = l \), then \( \lim_{n \to \infty} \sqrt{2a_n} = \sqrt{2l} \), by Theorem 1.

6. Let \( 0 < a_1 < b_1 \) and define
\[
a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{a_n + b_n}{2}.
\]
(a) Prove that the sequences \( \{a_n\} \) and \( \{b_n\} \) each converge.
(b) Prove that they have the same limit.

choice one - do all parts.

Section 2

1. Decide whether each of the following infinite series is convergent or divergent. The tools which you will need are Leibniz's Theorem and the comparison, ratio, and integral tests. A few examples have been picked with malice aforesight; two series which look quite similar may require different tests (and then again, they may not). The hint below indicates which tests may be used.

(i) \[ \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2} \]
(ii) \[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \]
(iii) \[ 1 - \frac{1}{2} + \frac{2}{3} - \frac{1}{4} + \frac{3}{5} - \frac{1}{6} + \frac{2}{7} + \cdots \]
(iv) \[ \sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n} \]
(v) \[ \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2 - 1}} \] (The summation begins with \( n = 2 \) simply to avoid the meaningless term obtained for \( n = 1 \).)
Hi. Use the comparison test for (i), (ii), (x), (xi), (xii), (xiii), (xiv), (xv), the ratio test for (xvi), (xvii), (xviii), (xix), the integral test for (viii), (x), (xx).

For each test, choose at least one.


1. Find the Taylor polynomials (of the indicated degree, and at the indicated point) for the following functions.

(i) \( f(x) = e^x \); degree 3, at 0.

(ii) \( f(x) = e^{\sin x} \); degree 3, at 0.

(iii) \( \sin x \); degree 2\(\pi\), at \(\frac{\pi}{2}\).

(iv) \( \cos x \); degree 2\(\pi\), at \(\pi\).

(v) \( \exp x \); degree \(n\), at 1.

(vi) \( \log x \); degree \(n\), at 2.

(vii) \( f(x) = x^6 + x^5 + x \); degree 4, at 0.

(viii) \( f(x) = x^8 + x^7 + x \); degree 4, at 1.

(ix) \( f(x) = \frac{1}{1 + x^2} \); degree 2\(\pi\) + 1, at 0.

(x) \( f(x) = \frac{1}{1 + x} \); degree \(n\), at 0.

Choose four. In each case, estimate the remainder on an interval of radius 0.2 about the center.

2. Write each of the following polynomials in \(x\) as a polynomial in \((x - 3)\). (It is only necessary to compute the Taylor polynomial at 3, of the same degree as the original polynomial. Why?)

(i) \( x^8 - 4x - 9 \).

(ii) \( x^4 - 12x^3 + 44x^2 + 2x + 1 \).

(iii) \( x^5 \).

(iv) \( ax^2 + bx + c \).