Finite Element Methods for Geometric Evolution Equations*

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Abstract. We study finite element methods for the solution of evolution equations in Riemannian geometry. Our focus is on Ricci flow and Ricci-DeTurck flow in two dimensions, where one of the main challenges from a numerical standpoint is to discretize the scalar curvature of a time-dependent Riemannian metric with finite elements. We propose a method for doing this which leverages Regge finite elements – piecewise polynomial symmetric $(0,2)$-tensors possessing continuous tangential-tangential components across element interfaces. In the lowest order setting, the finite element method we develop for two-dimensional Ricci flow is closely connected with a popular discretization of Ricci flow in which the scalar curvature is approximated with the so-called angle defect: 2π minus the sum of the angles between edges emanating from a common vertex. We present some results from our ongoing work on the analysis of the method, and we conclude with numerical examples.

Keywords: Finite element · Ricci flow · Scalar curvature · Angle defect

1 Introduction

Partial differential equations governing the evolution of time-dependent Riemannian metrics are ubiquitous in geometric analysis. In this work, we study finite element discretizations of such problems.

The model problem we consider consists of finding a Riemannian metric $g(t)$ on a smooth manifold Ω satisfying

$$\frac{\partial}{\partial t} g = \sigma, \quad g(0) = g_0, \tag{1}$$

where g_0 is given and σ is a symmetric $(0,2)$-tensor field depending on g and/or t. We are particularly interested in two special cases: (i) two-dimensional normalized Ricci flow, in which case $\sigma = (\bar{R} - R)g$, R is the scalar curvature of g, and $\bar{R}$ is the average of R over Ω (or some other prescribed scalar function); and (ii) two-dimensional Ricci-DeTurck flow, in which case $\sigma = -Rg + \mathcal{L}_w g$ and w is a certain vector field depending on g.

In both Ricci flow and Ricci-DeTurck flow, the problem can be recast as a coupled system of differential equations by treating the (densitized) scalar

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curvature R and the metric g as independent variables. As we show below, the system reads

$$\frac{\partial}{\partial t}(R\mu) = (\operatorname{div}_g \operatorname{div}_g S_g\sigma)\mu, \qquad R(0) = R_0, \tag{2}$$

$$\frac{\partial}{\partial t} g = \sigma, \qquad g(0) = g_0, \tag{3}$$

where R_0 is the scalar curvature of g_0, div_g is the covariant divergence operator, $\mu = \mu(g)$ is the volume form on Ω determined by g, and $(S_g\sigma)_{ij} = \sigma_{ij} - g_{ij}g^{k\ell}\sigma_{k\ell}$. An advantage of this formulation is that it eliminates the need to discretize the scalar curvature operator (the nonlinear second-order differential operator sending g to R). The scalar curvature R is instead initialized at $t = 0$ and evolved forward in time by solving the differential equation (2). The latter equation involves a differential operator $\operatorname{div}_g \operatorname{div}_g$ which is somewhat easier to discretize.

To fix ideas, let us consider the setting in which Ω is a 2-torus. Let $\mathcal{T}_h$ be a triangulation of Ω with maximum element diameter h. Assume that $\mathcal{T}_h$ belongs to a shape-regular, quasi-uniform family of triangulations parametrized by h. Let $\mathcal{E}_h$ denote the set of edges of $\mathcal{T}_h$. Let $q \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Define finite element spaces

$$V_h = \{v \in H^1(\Omega) \mid v|_K \in \mathcal{P}_q(K), \forall K \in \mathcal{T}_h\},$$
$$\Sigma_h = \{\sigma \in L^2(\Omega) \otimes \mathbb{S} \mid \sigma|_K \in \mathcal{P}_r(K) \otimes \mathbb{S}, \forall K \in \mathcal{T}_h, \text{ and } [\![\tau^T\sigma\tau]\!] = 0, \forall e \in \mathcal{E}_h\},$$

where $\mathcal{P}_r(K)$ denotes the space of polynomials of degree $\leq r$ on K, $[\![\tau^T\sigma\tau]\!]$ denotes the jump in the tangential-tangential component of σ across an edge $e \in \mathcal{E}_h$, and $\mathbb{S} = \{\sigma \in \mathbb{R}^{2\times 2} \mid \sigma = \sigma^T\}$. The space Σ_h is the space of *Regge finite elements* of degree r [13,4].

For scalar fields u and v on Ω, denote $\langle u, v\rangle_g = \int_\Omega uv\,\mu(g)$. For symmetric $(0,2)$-tensor fields σ and ρ defined on $K \in \mathcal{T}_h$, let $\langle \sigma, \rho\rangle_{g,K} = \int_K g^{ij}\sigma_{jk}g^{k\ell}\rho_{\ell i}\,\mu(g)$. For $e \in \mathcal{E}_h$, denote $\langle u, v\rangle_{g,e} = \int_e uv\sqrt{\tau^T g\tau}\,d\ell$, where τ is the unit vector tangent to e relative to the Euclidean metric δ, and $d\ell$ is the Euclidean line element along e. With $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, let $\tau_g = \tau/\sqrt{\tau^T g\tau}$, $n_g = Jg\tau/\sqrt{\tau^T g\tau \det g}$, and $\frac{\partial v}{\partial n_g} = n_g^T g\nabla_g v$. Let $\operatorname{Hess}_g v$ denote the Riemannian Hessian of v.

To discretize the operator $\operatorname{div}_g \operatorname{div}_g S_g$ appearing in (2), we make use of the metric-dependent bilinear form

$$b_h(g;\sigma,v) = \sum_{K\in\mathcal{T}_h} \langle S_g\sigma, \operatorname{Hess}_g v\rangle_{g,K} + \sum_{e\in\mathcal{E}_h} \left\langle \tau_g^T\sigma\tau_g, \left[\!\!\left[\frac{\partial v}{\partial n_g}\right]\!\!\right]\right\rangle_{g,e}.$$

This bilinear form is a non-Euclidean generalization of the bilinear form used in the classical Hellan-Herrmann-Johnson mixed discretization of the biharmonic equation [9, p. 237]. Using integration by parts, it can be shown that for smooth g, σ, and v, we have $b_h(g;\sigma,v) = \int_\Omega (\operatorname{div}_g \operatorname{div}_g S_g\sigma)v\,\mu(g)$.

To discretize (2-3), we choose approximations $R_{h0} \in V_h$ and $g_{h0} \in \Sigma_h$ of R_0 and g_0, respectively. We then seek $R_h(t) \in V_h$ and $g_h(t) \in \Sigma_h$ such that

$R_h(0) = R_{h0}$, $g_h(0) = g_{h0}$, and

$$\frac{\partial}{\partial t}\langle R_h, v_h\rangle_{g_h} = b_h(g_h; \sigma_h, v_h), \qquad \forall v_h \in V_h, \tag{4}$$

$$\frac{\partial}{\partial t} g_h = \sigma_h, \tag{5}$$

where $\sigma_h = \sigma_h(g_h, R_h, t)$ is a discretization of σ. For the moment, we postpone discussing our choice of σ_h; this will be addressed in the next sections. We assume throughout what follows that (1) and (4-5) preserve the signature of g and g_h, in the sense that the eigenvalues of g and g_h are bounded from below by a positive constant independent of h, x, and t.

1.1 Connection with the angle defect

An important feature of (4) is its connection with the widely studied *angle defect* from discrete differential geometry [2,14,5]. Recall that the angle defect Θ_i at the i^{th} vertex $y^{(i)} \in \Omega$ of the triangulation $\mathcal{T}_h$ measures the failure of the angles incident at $y^{(i)}$ to sum up to 2π:

$$\Theta_i = 2\pi - \sum_{K\in\omega_i} \theta_{iK}. \tag{6}$$

Here, ω_i denotes the set of triangles in $\mathcal{T}_h$ having $y^{(i)}$ as a vertex, and θ_{iK} denotes the interior angle of K at $y^{(i)}$. The following proposition shows that in the lowest order setting ($r = 0$ and $q = 1$), the differential equation (4) reproduces the angle defect if R_{h0} is chosen appropriately.

Proposition 1. *Let $r = 0$ and $q = 1$. Let $\{\phi_i\}_i$ be the basis for V_h satisfying $\phi_i(y^{(j)}) = \delta_i^j$, and let Θ_{i0} be the angle defect at vertex $y^{(i)}$ as measured by g_{h0}. If*

$$\langle R_{h0}, \phi_i\rangle_{g_{h0}} = 2\Theta_{i0}, \tag{7}$$

then the solution of (4)-(5) satisfies

$$\langle R_h(t), \phi_i\rangle_{g_h(t)} = 2\Theta_i(t)$$

for every t, where $\Theta_i(t)$ is the angle defect at vertex $y^{(i)}$ as measured by $g_h(t)$.

Proof. It is shown in [8, Lemma 3.3] that

$$\frac{\partial}{\partial t}(2\Theta_i(t)) = b_h\left(g_h(t); \frac{\partial}{\partial t} g_h(t), \phi_i\right), \tag{8}$$

so $2\Theta_i(t)$ and $\langle R_h(t), \phi_i\rangle_{g_h(t)}$ obey the same ordinary differential equation.

The relation (8) is a discrete analogue of the following relation which holds in the smooth setting.

Proposition 2. *Let $g(t)$ be a smooth Riemannian metric on Ω depending smoothly on t. Then, for every smooth scalar field v,*

$$\frac{\partial}{\partial t}\langle R(g(t)), v\rangle_{g(t)} = \left\langle \operatorname{div}_{g(t)} \operatorname{div}_{g(t)} S_{g(t)} \frac{\partial g}{\partial t}, v \right\rangle_{g(t)}.$$

Remark 1. The relation above is not valid in dimensions greater than 2.

Proof. We have

$$\frac{\partial}{\partial t}\langle R(g(t)), v\rangle_{g(t)} = \int_\Omega (DR(g(t)) \cdot \sigma(t))\, v\, \mu(g(t)) + \int_\Omega R(g(t)) v\, (D\mu(g(t)) \cdot \sigma(t)),$$

where $\sigma(t) = \frac{\partial}{\partial t} g(t)$. The linearizations of R and μ are given by [6, Lemma 2]

$$DR(g) \cdot \sigma = \operatorname{div}_g \operatorname{div}_g \sigma - \Delta_g(g^{ij}\sigma_{ij}) - g^{ij}\sigma_{jk} g^{k\ell} \operatorname{Ric}_{\ell i},$$
$$D\mu(g) \cdot \sigma = \frac{1}{2} g^{ij}\sigma_{ij}\mu(g).$$

Since $\operatorname{Ric} = \frac{1}{2}Rg$ in two dimensions and $\Delta_g u = \operatorname{div}_g \operatorname{div}_g(gu)$ for any scalar field u, the first expression simplifies to

$$DR(g) \cdot \sigma = \operatorname{div}_g \operatorname{div}_g S_g\sigma - \frac{1}{2} R g^{ij}\sigma_{ij}.$$

Combining these gives

$$\frac{\partial}{\partial t}\langle R(g(t)), v\rangle_{g(t)} = \int_\Omega (\operatorname{div}_g \operatorname{div}_g S_g\sigma)\, v\, \mu.$$

2 Ricci flow

Let us now focus on two-dimensional normalized Ricci flow, which corresponds to the choice $\sigma = (\bar{R} - R)g$ in (1). As before, R is the scalar curvature of g and $\bar{R}$ is the average of R over Ω (or some other prescribed scalar function).

Several simplifications can be made in this setting. Since σ is proportional to g, we have $\operatorname{div}_g \operatorname{div}_g S_g\sigma = \Delta_g(\bar{R} - R) - 2\Delta_g(\bar{R} - R) = \Delta_g(R - \bar{R})$, so that (2) reduces to

$$\frac{\partial}{\partial t}(R\mu) = (\Delta_g(R - \bar{R}))\mu.$$

This offers us some flexibility in our choice of discretization. One option is to use (4-5) as it is written, choosing σ_h equal to

$$\sigma_h = P_h((\bar{R}_h - R_h)g_h), \tag{9}$$

where P_h is any projector onto Σ_h whose domain contains $\{v_h\rho_h \mid v_h \in V_h, \rho_h \in \Sigma_h\}$, and $\bar{R}_h \in V_h$ is equal to $\bar{R}$ or an approximation thereof. Another option is to use the discretization

$$\frac{\partial}{\partial t}\langle R_h, v_h\rangle_{g_h} = \langle \nabla_{g_h}(\bar{R}_h - R_h), \nabla_{g_h} v_h\rangle_{g_h}, \qquad \forall v_h \in V_h, \tag{10}$$

$$\frac{\partial}{\partial t} g_h = \sigma_h, \tag{11}$$

again with σ_h given by (9).

The next proposition gives an example of a setting in which (4-5) and (10-11) are equivalent. In it, we denote by $z^{(e)} \in \Omega$ the midpoint of an edge $e \in \mathcal{E}_h$. Note that when $r = 0$, the linear functionals

$$\rho \mapsto \tau^T \rho(z^{(e)})\tau, \quad e \in \mathcal{E}_h$$

form a basis for the dual of Σ_h. We denote by $\{\psi_e\}_{e\in\mathcal{E}_h} \subset \Sigma_h$ the basis for Σ_h satisfying

$$\tau^T \psi_e(z^{(e')})\tau = \begin{cases} 1, & \text{if } e = e', \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3. *Let $r = 0$ and $q = 1$. Let P_h be given by*

$$P_h \rho = \sum_{e\in\mathcal{E}_h} (\tau^T \rho(z^{(e)})\tau)\psi_e,$$

and let σ_h be given by (9). Choose R_{h0} equal to the unique element of V_h satisfying (7) for every i. Then, with initial conditions $R_h(0) = R_{h0}$ and $g_h(0) = g_{h0}$, problems (4-5) and (10-11) are equivalent. Furthermore, the solution $g_h(t)$ satisfies

$$g_h(t) = P_h(e^{u_h(t)} g_{h0}), \tag{12}$$

where $u_h(t) \in V_h$ obeys the differential equation

$$\frac{\partial}{\partial t} u_h = \bar{R}_h - R_h, \quad u_h(0) = 0, \tag{13}$$

and the solution $R_h(t)$ satisfies

$$\langle R_h(t), \phi_i \rangle_{g_h(t)} = 2\Theta_i(t) \tag{14}$$

for every t and every i, where $\Theta_i(t)$ is the angle defect at vertex $y^{(i)}$ as measured by $g_h(t)$.

Proof. Using the fact that functions in V_h are piecewise linear when $q = 1$, one verifies through integration by parts that

$$\begin{aligned} b_h(g_h; P_h((\bar{R}_h - R_h)g_h), v_h) &= b_h(g_h; (\bar{R}_h - R_h)g_h, v_h) \\ &= \langle \nabla_{g_h}(\bar{R}_h - R_h), \nabla_{g_h} v_h \rangle_{g_h} \end{aligned}$$

for every $v_h \in V_h$. This demonstrates the equivalence of (4-5) and (10-11). To deduce (12-13), observe that differentiating (12) and invoking (13) gives

$$\begin{aligned} \frac{\partial}{\partial t} g_h &= P_h\left((\bar{R}_h - R_h)e^{u_h} g_{h0}\right) \\ &= P_h\left((\bar{R}_h - R_h)P_h(e^{u_h} g_{h0})\right) \\ &= P_h\left((\bar{R}_h - R_h)g_h\right) \\ &= \sigma_h \end{aligned}$$

where the second line above follows from our choice of P_h. The relation (14) between $R_h(t)$ and the angle defect follows from Proposition 1.

2.1 Connection with other discretizations of Ricci flow

Proposition 3 reveals a close connection between the lowest-order version of our finite element discretization of Ricci flow and another popular finite difference scheme for Ricci flow [3,11]. In this popular method, (Ω, g) is discretized with a triangulation having time-dependent edge lengths ℓ_{ij} between adjacent vertices i and j. The scalar curvature $R(g)$ (which is twice the Gaussian curvature) is then approximated by (two times) the angle defect. The method stores a time-dependent scalar u_i at each vertex i which evolves according to

$$\frac{\partial}{\partial t} u_i = 2(\bar{\Theta}_i - \Theta_i). \tag{15}$$

where $\bar{\Theta}_i$ is prescribed. (Note that in [3], (15) is expressed in terms of $r_i := e^{u_i/2}$ rather than u_i.) This collection of scalars determines the lengths ℓ_{ij} of all edges at time t in terms of their lengths at $t = 0$ via a relation which is analogous to (12) but is motivated by circle packing theory [12] rather than finite element theory. (Other choices are also possible; see [10, Section 5] and [15] for a discussion of alternatives.)

The connection with our finite element discretization is now transparent. In the lowest order instance of our finite element discretization ($r = 0$ and $q = 1$), the degrees of freedom for u_h and g_h are the values of u_h at each vertex and the squared length of each edge as measured by g_h. According to equations (13) and (12), these degrees of freedom evolve in nearly the same way that u_i and ℓ_{ij} evolve in [3,11].

There is one important discrepancy, however: Equation (15) is not a consistent discretization of normalized Ricci flow. This is because the angle defect (6) approximates the *integral* of the Gaussian curvature over a cell which is dual to vertex i, not its average over the cell. See [1, Remark B.2.4] for more insight. In many applications, this is not a serious concern, since very often the goal is not to accurately approximate Ricci flow, but rather to construct a discrete conformal mapping from a given triangulation to one with prescribed discrete curvature.

Putting this discrepancy aside, the similarities noted above suggest that our finite element method (with $r \geq 0$ and $q \geq 1$) can be loosely regarded as a high-order generalization of the scheme studied in [3,11]. A link like this does not appear to hold for some other finite element discretizations of Ricci flow such as the one studied in [7]. In particular, [7] relies on the existence of an embedding of (Ω, g) into $\mathbb{R}^3$.

3 Error Analysis

We now discuss some of our ongoing work on the analysis of the accuracy of the discretization (4-5). One setting which is particularly easy to analyze is that in which σ and σ_h are prescribed functions of t. Then estimates for $g_h - g$ are immediate, and it remains to estimate $R_h - R$. The following proposition gives estimates for $R_h - R$ in the metric-dependent negative-order Sobolev-norm (recall that Ω has no boundary)

$$\|v\|_{H^{-1}(\Omega,g)} = \sup_{u \in H^1(\Omega)} \frac{\langle v, u \rangle_g}{\|u\|_{H^1(\Omega)}}. \tag{16}$$

In what follows, we take $\bar{R} = \bar{R}_h$ to be constant, we assume $r > 0$, and we make use of the broken Sobolev semi-norm $|\sigma|_{H^1_h(\Omega)} = \left(\sum_{K \in \mathcal{T}_h} |\sigma|^2_{H^1(K)} \right)^{1/2}$.

Proposition 4. *If σ and σ_h depend only on t, and if g and R are sufficiently regular, then for $T > 0$ small enough, the solutions of (2-3) and (4-5) satisfy*

$$\|g_h(T) - g(T)\|_{L^2(\Omega)} \leq \|g_{h0} - g_0\|_{L^2(\Omega)} + \int_0^T \|\sigma_h(t) - \sigma(t)\|_{L^2(\Omega)} \, dt,$$

$$\begin{aligned}
&\|R_h(T) - R(T)\|_{H^{-1}(\Omega, g(T))} \\
&\quad \leq C\bigg(\int_0^T \left(h^{-1} \|\sigma_h(t) - \sigma(t)\|_{L^2(\Omega)} + |\sigma_h(t) - \sigma(t)|_{H^1_h(\Omega)} \right) dt \\
&\qquad\qquad + \inf_{u_h \in V_h} \|R(T) - u_h\|_{H^{-1}(\Omega, g(T))} + \|R_{h0} - R_0\|_{H^{-1}(\Omega, g(T))} \bigg).
\end{aligned}$$

Proof. The estimate for $g_h(T) - g(T)$ is immediate, and the estimate for $R_h(T) - R(T)$ can be obtained by extending the analysis in [8], which studies the case in which $g(t) = \frac{T-t}{T}\delta + \frac{t}{T} g(T)$, $g_h(t) = \frac{T-t}{T}\delta + \frac{t}{T} g_h(T)$, and $R_{h0} = R_0 = 0$.

Choosing g_{h0}, R_{h0}, and $\sigma_h(t)$ equal to suitable interpolants of g_0, R_0, and $\sigma(t)$, one obtains from Proposition 4 estimates of the form

$$\begin{aligned}
\|g_h(T) - g(T)\|_{L^2(\Omega)} &\leq C h^{r+1}, \\
\|R_h(T) - R(T)\|_{H^{-1}(\Omega, g(T))} &\leq C(h^r + h^{q+2})
\end{aligned}$$

for sufficiently regular solutions.

4 Numerical Examples

Figure 1 plots a numerical simulation of normalized Ricci flow obtained using the finite element method (4-5) with the parameter choices described in Proposition 3. Here, $\mathcal{T}_h$ was taken equal to a triangulation of a 2-sphere rather than a 2-torus. At each instant $t \geq 0$, we visualized $g_h(t)$ by numerically determining an embedding of the vertices of $\mathcal{T}_h$ into $\mathbb{R}^3$ with the property that the distances between adjacent vertices agree with the edge lengths determined by $g_h(t)$.

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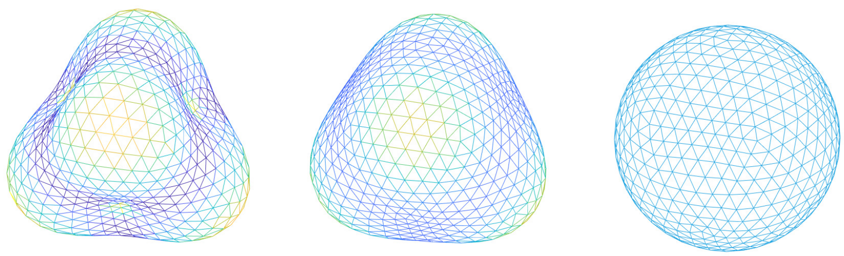

Fig. 1. Numerical solution at $t = 0$, $t = 0.05$, and $t = 0.75$.

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