ZOLOTAREV'S FIFTH AND SIXTH PROBLEMS

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Abstract. In an influential 1877 paper, Zolotarev asked and answered four questions about polynomial and rational approximation. We ask and answer two questions: what are the best rational approximants r and s to $\sqrt{z}$ and $\operatorname{sign}(z)$ on the unit circle (excluding certain arcs near the discontinuities), with the property that $|r(z)| = |s(z)| = 1$ for $|z| = 1$? We show that the solutions to these problems are related to Zolotarev's third and fourth problems in a nontrivial manner.

1. Introduction

Nearly 150 years ago, Zolotarev asked and answered four questions from approximation theory [23]. The first two concern polynomial approximation. The third is equivalent to the fourth, and the fourth concerns the approximation of $\operatorname{sign}(x) = x/\sqrt{x^2}$ by rational functions on $[-1, -\ell] \cup [\ell, 1]$, where $\ell \in (0, 1)$. His solutions to these problems have had lasting impact in approximation theory [1, 9, 19, 21] and numerical analysis [2, 6, 10, 11, 13, 16, 20, 22].

In this paper, we ask and answer two questions that are closely related to Zolotarev's fourth problem: what are the best (in the uniform norm) rational approximants r and s to $\sqrt{z}$ and $\operatorname{sign}(z) = z/\sqrt{z^2}$ on the unit circle (excluding certain arcs near the discontinuities), with the property that $|r(z)| = |s(z)| = 1$ for $|z| = 1$? We derive explicit solutions to these two problems and show that they are related in a nontrivial manner to the solution of Zolotarev's fourth problem. We also show a remarkable property of these solutions: composing two best rational approximants of $\operatorname{sign}(z)$ on the unit circle yields a best rational approximant of higher degree. This phenomenon closely mirrors the behavior of best rational approximants of $\operatorname{sign}(x)$ on $[-1, -\ell] \cup [\ell, 1]$ [3, 4, 16]. Related composition laws for best rational approximants have appeared in other contexts, such as the approximation of the square root and pth root on positive real intervals [6–8] and the solution of certain extremal problems involving finite Blaschke products [17, 18]. Some other rational approximation problems on the unit circle have been studied in [14, 15].

Let us give precise statements of the problems that we study, beginning with some notation. We say that a rational function $r(z) = p(z)/q(z)$ has type (m, n) if p and q are polynomials of degree at most m and n, respectively. We use $\mathcal{R}_{m,n}$ to denote the set of rational functions of type (m, n) with complex coefficients, and $\mathcal{R}_{m,n}^{\text{real}}$ to denote the set of rational functions of type (m, n) with real coefficients. We say that $r \in \mathcal{R}_{m,n}$ has *exact type* (μ, ν) if, after canceling common factors, p

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and q have degree exactly μ and ν, respectively. For each $\Theta \in (0,\pi/2)$, we let

$$S_\Theta = \{e^{i\theta} \colon \theta \in [-2\Theta, 2\Theta]\},$$
$$T_\Theta = \{e^{i\theta} \colon \theta \in [-\Theta,\Theta] \cup [\pi-\Theta, \pi+\Theta]\}.$$

We address the following rational approximation problems.

Problem Z5. Given $\Theta \in (0,\pi/2)$ and $n \in \mathbb{N}_0$, find the rational function in $\{r \in \mathcal{R}_{n,n} \colon |r(z)| = 1 \text{ on } |z| = 1\}$ that minimizes

$$\max_{z\in S_\Theta} \left| \arg\left(\frac{r(z)}{\sqrt{z}} \right) \right|.$$

Problem Z6. Given $\Theta \in (0,\pi/2)$ and $m \in \mathbb{N}_0$, find the rational function in $\{r \in \mathcal{R}_{m,m} \colon |r(z)| = 1 \text{ on } |z| = 1\}$ that minimizes

$$\max_{z\in T_\Theta} \left| \arg\left(\frac{r(z)}{\operatorname{sign}(z)} \right) \right|.$$

We labeled the above problems "Z5" and "Z6" since they are natural follow-ups to Zolotarev's fourth problem, which we label "Z4". Zolotarev's fourth problem reads as follows (up to a trivial scaling).

Problem Z4. Given $\ell \in (0,1)$ and $m \in \mathbb{N}_0$, find the rational function $r \in \mathcal{R}^{\mathrm{real}}_{m,m}$ that minimizes

$$\max_{x\in[-1,-\ell]\cup[\ell,1]} |r(x) - \operatorname{sign}(x)|.$$

We will derive explicit solutions to Problems Z5 and Z6 in Section 2. As we shall see, the solutions to problems Z4–Z6 are connected in a nontrivial manner. Then we will study their properties in Section 3, including their behavior under composition and their error.

2. Solutions

In this section, we derive explicit solutions to Problems Z5 and Z6. The solutions, summarized in Theorem 2.1, involve Jacobi's elliptic functions. We use $\operatorname{sn}(\cdot,\ell)$, $\operatorname{cn}(\cdot,\ell)$, and $\operatorname{dn}(\cdot,\ell)$ to denote Jacobi's elliptic functions with modulus ℓ, and we use $\ell' = \sqrt{1-\ell^2}$ to denote the modulus complementary to ℓ. We denote the complete elliptic integral of the first kind by $K(\ell) = \int_0^{\pi/2} (1-\ell^2\sin^2\theta)^{-1/2}\, d\theta$.

Theorem 2.1. *Let $m,n \in \mathbb{N}_0$ and $\Theta \in (0,\pi/2)$. Problem Z5 has a unique solution given by*

$$r(z) = r_n(z;\Theta) = \prod_{j=1}^{n} \frac{1+a_j z}{z+a_j}, \tag{2.1}$$

where

$$a_j = \left(\frac{\ell \operatorname{sn}\left(\frac{2j-1}{2n+1}K(\ell'),\ell'\right) + \operatorname{dn}\left(\frac{2j-1}{2n+1}K(\ell'),\ell'\right)}{\operatorname{cn}\left(\frac{2j-1}{2n+1}K(\ell'),\ell'\right)} \right)^{2(-1)^{j+n}}, \tag{2.2}$$

and $\ell = \cos\Theta$. Problem Z6 has two solutions: the function

$$s(z) = s_m(z;\Theta) = i^{1-m} \prod_{j=1}^{m} \frac{z - ib_j}{1 + ib_j z}, \tag{2.3}$$

and its reciprocal, where

$$b_j = (-1)^{mj} \left(\frac{\ell \operatorname{sn}\left(\frac{2j-1}{m} K(\ell'), \ell'\right) + \operatorname{dn}\left(\frac{2j-1}{m} K(\ell'), \ell'\right)}{\operatorname{cn}\left(\frac{2j-1}{m} K(\ell'), \ell'\right)} \right)^{(-1)^j}. \tag{2.4}$$

Remark 2.2. When $m = 2n+1$, the functions $s_m(z;\Theta)$ and $r_n(z;\Theta)$ are related to one another. Using the observation that

$$b_j = \begin{cases} (-1)^j \sqrt{a_j}, & \text{if } m = 2n+1,\ j < n+1, \text{ and } n \text{ is even,} \\ 0, & \text{if } m = 2n+1,\ j = n+1, \text{ and } n \text{ is even,} \\ (-1)^{j+1} \sqrt{a_{2n+2-j}}, & \text{if } m = 2n+1,\ j > n+1, \text{ and } n \text{ is even,} \\ (-1)^j / \sqrt{a_j}, & \text{if } m = 2n+1,\ j < n+1, \text{ and } n \text{ is odd,} \\ \infty, & \text{if } m = 2n+1,\ j = n+1, \text{ and } n \text{ is odd,} \\ (-1)^{j+1} / \sqrt{a_{2n+2-j}}, & \text{if } m = 2n+1,\ j > n+1, \text{ and } n \text{ is odd,} \end{cases}$$

one checks that

$$s_{2n+1}(z;\Theta)^{(-1)^n} = z \prod_{j=1}^{n} \frac{z^2 + a_j}{1 + a_j z^2} = \frac{z}{r_n(z^2;\Theta)}. \tag{2.5}$$

In particular, $s_{2n+1}(z;\Theta)$ has exact type $(2n+1, 2n)$ when n is even, and it has exact type $(2n, 2n+1)$ when n is odd. On the other hand, $s_{2n}(z;\Theta)$ has exact type $(2n, 2n)$.

Remark 2.3. For $m, n > 1$, neither $s_m(z;\Theta)$ nor $r_n(z;\Theta)$ is a finite Blaschke product, since both functions have at least one root outside the unit disk.

The following identity will play a central role in our proof of Theorem 2.1.

Theorem 2.4. *Let $\ell \in (0,1)$ and $m \in \mathbb{N}$. Let $M = K(\ell)/K(\lambda)$, where $\lambda \in (0,1)$ is determined uniquely by the condition that*

$$\frac{K(\ell)}{K(\ell')} = \frac{K(\lambda)}{mK(\lambda')} \tag{2.6}$$

holds with $\lambda' = \sqrt{1-\lambda^2}$. Then the function $s(z)$ in (2.3) *can be expressed as*

$$s(z) = F(x) + i \operatorname{sign}(\operatorname{Im} z)^m G(x), \quad x = \frac{1}{2}(z + z^{-1}), \tag{2.7}$$

where

$$F(x) = F_m(x;\ell) = \lambda \operatorname{sn}\left(\frac{\operatorname{sn}^{-1}(x/\ell, \ell)}{M}, \lambda \right), \tag{2.8}$$

$$G(x) = G_m(x;\ell) = \operatorname{dn}\left(\frac{\operatorname{sn}^{-1}(x/\ell, \ell)}{M}, \lambda \right). \tag{2.9}$$

The function $F(x)$ appearing above is none other than Zolotarev's classical solution to Problem Z4 on $[-1,-\ell] \cup [\ell, 1]$, scaled to have maximum value 1 on $[\ell, 1]$ ([23], [1, Sections 50-51]):

$$\frac{2}{1+\lambda} F = \operatorname*{arg\,min}_{r \in \mathcal{R}^{\text{real}}_{m,m}} \max_{x \in [-1,-\ell] \cup [\ell,1]} |r(x) - \operatorname{sign}(x)|.$$

It is well-known that $F(x)$ is an odd rational function of exact type $(2\lfloor (m-1)/2 \rfloor + 1, 2\lfloor m/2 \rfloor)$ that is real-valued on $\mathbb{R}$ and oscillates between λ and 1 on $[\ell, 1] = [\cos\Theta, 1]$, achieving these values at $m+1$ points $\ell = x_0 < x_1 < \cdots < x_m = 1$ in an alternating fashion ([2, p. 9], [1, Sections 50-51]). In particular, $F(\ell) = \lambda$. Since $|s(z)| = 1$ for $|z| = 1$, it follows from (2.7) that $\arg(s(e^{i\theta}))$ equioscillates $m+1$ times on $[-\Theta, \Theta]$, taking values in $[-\arccos\lambda, \arccos\lambda]$. That is,

(2.10)
$$\arg(s(e^{i\theta_j})) = \sigma(-1)^j \max_{\theta\in[-\Theta,\Theta]} |\arg(s(e^{i\theta}))| = \sigma(-1)^j \arccos\lambda, \quad j = 0, 1, \dots, m,$$

where $\sigma \in \{-1, 1\}$ and

$$\theta_j = \begin{cases} -\arccos x_{2j}, & \text{if } j \le m/2, \\ \arccos x_{2m-2j}, & \text{if } j > m/2. \end{cases}$$

We will eventually use this fact, together with Remark 2.2, to prove the optimality of s and r.

2.1. **Proof of Theorem 2.4.** Let us first prove Theorem 2.4, beginning with the case in which $m = 2n+1$ and n is even.

Case 1 ($m = 2n+1$, n even). The fact that the right-hand side of (2.7) is a rational function of z, much less of type $(2n+1, 2n)$, is not obvious at first glance. To prove this, we recall the identities [1, p. 214]

$$\operatorname{sn}\left(\frac{u}{M}, \lambda\right) = \frac{\operatorname{sn}(u,\ell)}{M} \prod_{k=1}^{n} \frac{1 + \operatorname{sn}^2(u,\ell)\frac{\operatorname{cn}^2(v_{2k},\ell')}{\operatorname{sn}^2(v_{2k},\ell')}}{1 + \operatorname{sn}^2(u,\ell)\frac{\operatorname{cn}^2(v_{2k-1},\ell')}{\operatorname{sn}^2(v_{2k-1},\ell')}}, \tag{2.11}$$

$$\operatorname{dn}\left(\frac{u}{M}, \lambda\right) = \operatorname{dn}(u,\ell) \prod_{k=1}^{n} \frac{1 - \operatorname{sn}^2(u,\ell)\operatorname{dn}^2(v_{2k-1},\ell')}{1 + \operatorname{sn}^2(u,\ell)\frac{\operatorname{cn}^2(v_{2k-1},\ell')}{\operatorname{sn}^2(v_{2k-1},\ell')}}, \tag{2.12}$$

where $v_j = \frac{j}{m}K(\ell')$. Let us denote

$$\widetilde{F}(z) = \widetilde{F}_{2n+1}(z;\Theta) = F_{2n+1}\left(\frac{1}{2}(z+z^{-1});\ell\right), \tag{2.13}$$

$$\widetilde{G}(z) = \widetilde{G}_{2n+1}(z;\Theta) = \operatorname{sign}(\operatorname{Im} z) G_{2n+1}\left(\frac{1}{2}(z+z^{-1});\ell\right). \tag{2.14}$$

Note that $\widetilde{F}(z) - i\widetilde{G}(z) = (\widetilde{F}(z) + i\widetilde{G}(z))^{-1}$ since $\lambda^2\operatorname{sn}^2(\cdot,\lambda) + \operatorname{dn}^2(\cdot,\lambda) = 1$. Using the fact that

$$\ell\operatorname{sn}(u,\ell) = \frac{1}{2}(z+z^{-1}) \iff \operatorname{dn}(u,\ell) = \frac{1}{2i}(z-z^{-1})\operatorname{sign}(\operatorname{Im} z),$$

we can write

$$\widetilde{F}(z) = \frac{\lambda}{2M\ell}(z+z^{-1}) \prod_{k=1}^{n} \frac{1 + \frac{\left(\frac{1}{2}(z+z^{-1})\right)^2 \operatorname{cn}^2(v_{2k},\ell')}{\ell^2\operatorname{sn}^2(v_{2k},\ell')}}{1 + \frac{\left(\frac{1}{2}(z+z^{-1})\right)^2 \operatorname{cn}^2(v_{2k-1},\ell')}{\ell^2\operatorname{sn}^2(v_{2k-1},\ell')}},$$

$$\widetilde{G}(z) = \frac{1}{2i}(z-z^{-1}) \prod_{k=1}^{n} \frac{1 - \frac{\left(\frac{1}{2}(z+z^{-1})\right)^2}{\ell^2}\operatorname{dn}^2(v_{2k-1},\ell')}{1 + \frac{\left(\frac{1}{2}(z+z^{-1})\right)^2 \operatorname{cn}^2(v_{2k-1},\ell')}{\ell^2\operatorname{sn}^2(v_{2k-1},\ell')}}.$$

From these expressions it is easy to deduce that $\widetilde{F}(z) + i\widetilde{G}(z)$ is a rational function which is ostensibly of type $(4n+2, 4n+1)$. However, this turns out to be an overestimate: $\widetilde{F}(z)$ and $i\widetilde{G}(z)$ have $2n+1$ coincident poles (one of which is at $z = 0$) with opposite residues, rendering $\widetilde{F}(z) + i\widetilde{G}(z)$ of type $(2n+1, 2n)$.

To see why, it is helpful to rewrite $F(x)$ and $G(x)$ in terms of the Grötsch ring function

$$\mu(\lambda) = \frac{\pi}{2}\frac{K(\lambda')}{K(\lambda)}, \quad \lambda' = \sqrt{1-\lambda^2}$$

and the functions

$$\begin{aligned} f_\nu(x) &= \ell\,\mathrm{sn}(K(\ell)x,\ell), \\ g_\nu(x) &= \mathrm{dn}(K(\ell)x,\ell), \end{aligned} \qquad \ell = \mu^{-1}(1/\nu).$$

One readily checks, using (2.6), that

$$\begin{aligned} F(x) &= f_{m\nu}(f_\nu^{-1}(x)), \\ G(x) &= g_{m\nu}(f_\nu^{-1}(x)), \end{aligned} \qquad \nu = \frac{1}{\mu(\ell)}. \tag{2.15}$$

Similar formulas for F appear in [3, 4].

Next, we recall that the poles of $\mathrm{sn}(u,\lambda)$ occur at $u \in \{2pK(\lambda) + i(2j-1)K(\lambda') \mid p,j \in \mathbb{Z}\}$ [12, Equation 2.2.9]. The finite nonzero poles of $\widetilde{F}(z)$ thus occur at those $z \in \mathbb{C}$ for which

$$K(\lambda)f_\nu^{-1}\left(\frac{1}{2}(z+z^{-1})\right) = 2pK(\lambda) + i(2j-1)K(\lambda'), \quad p,j \in \mathbb{Z}. \tag{2.16}$$

That is,

$$\begin{aligned} \frac{1}{2}(z+z^{-1}) &= f_\nu\left(2p + i(2j-1)\frac{K(\lambda')}{K(\lambda)}\right) \\ &= f_\nu\left(2p + i\frac{2j-1}{m}\frac{K(\ell')}{K(\ell)}\right) \\ &= \ell\,\mathrm{sn}\left(2pK(\ell) + i\frac{2j-1}{m}K(\ell'),\ell\right) \\ &= (-1)^p\ell\,\mathrm{sn}(iv_{2j-1},\ell). \end{aligned}$$

Here, we used (2.6), the notation $v_j = \frac{j}{m}K(\ell')$, and the half-period identity $\mathrm{sn}(2pK(\ell)+u,\ell) = (-1)^p\,\mathrm{sn}(u,\ell)$ [12, Equation 2.2.11].

The numbers z satisfying $\frac{1}{2}(z+z^{-1}) = (-1)^p\ell\,\mathrm{sn}(iv_{2j-1},\ell)$ are given by

$$z = (-1)^p\left(\ell\,\mathrm{sn}(iv_{2j-1},\ell) \pm i\,\mathrm{dn}(iv_{2j-1},\ell)\right).$$

Indeed, since $\ell^2\,\mathrm{sn}^2(\cdot,\ell)+\mathrm{dn}^2(\cdot,\ell) = 1$, we have $z^{-1} = (-1)^p\left(\ell\,\mathrm{sn}(iv_{2j-1},\ell) \mp i\,\mathrm{dn}(iv_{2j-1},\ell)\right)$.

We conclude that the finite nonzero poles of $\widetilde{F}(z)$ occur at

$$\{z_{j,p,q} \mid p,q = 0,1,\ j = 1,2,\dots,n\},$$

where

$$z_{j,p,q} = (-1)^p\left(\ell\,\mathrm{sn}(iv_{2j-1},\ell) + (-1)^q i\,\mathrm{dn}(iv_{2j-1},\ell)\right).$$

The finite nonzero poles of $i\widetilde{G}(z)$ are identical, since $\mathrm{dn}(\cdot,\lambda)$ and $\mathrm{sn}(\cdot,\lambda)$ have the same poles. All of these poles are simple poles thanks to the simplicity of the poles of sn and dn.

Below we relate the residues of $\widetilde{F}(z)$ to those of $i\widetilde{G}(z)$.

Lemma 2.5. *We have*

$$\mathrm{Res}(\widetilde{F},\, z_{j,p,q}) = \begin{cases} \mathrm{Res}(i\widetilde{G},\, z_{j,p,q}), & \text{if } j+q \text{ is odd}, \\ -\mathrm{Res}(i\widetilde{G},\, z_{j,p,q}), & \text{if } j+q \text{ is even}. \end{cases} \tag{2.17}$$

In particular, $\mathrm{Res}(\widetilde{F}+i\widetilde{G},\, z_{j,p,q}) = 0$ *if* $j+q$ *is even.*

Proof. In view of (2.15), the residues of $F\left(\frac{1}{2}(z+z^{-1})\right)$ and $G\left(\frac{1}{2}(z+z^{-1})\right)$ at $z_{j,p,q}$ are proportional to the residues of $f_{m\nu}(u) = \lambda \operatorname{sn}(K(\lambda)u, \lambda)$ and $g_{m\nu}(u) = \operatorname{dn}(K(\lambda)u, \lambda)$ at $u = f_\nu^{-1}\left(\frac{1}{2}(z_{j,p,q} + z_{j,p,q}^{-1})\right) =: u_{j,p,q}$, with the constant of proportionality the same in both cases. From (2.16), we have

$$K(\lambda)u_{j,p,q} = 2pK(\lambda) + i(2j-1)K(\lambda'),$$

so [12, p. 41-42]

$$\operatorname{Res}(\lambda \operatorname{sn}(K(\lambda)u, \lambda),\, u_{j,p,q}) = (-1)^p / K(\lambda), \tag{2.18}$$

$$\operatorname{Res}(\operatorname{dn}(K(\lambda)u, \lambda),\, u_{j,p,q}) = (-1)^j i / K(\lambda). \tag{2.19}$$

Since $\operatorname{sign}(\operatorname{Im} z_{j,p,q}) = (-1)^{p+q}$, it follows that

$$i \operatorname{sign}(\operatorname{Im} z_{j,p,q}) \operatorname{Res}(\operatorname{dn}(K(\lambda)u, \lambda),\, u_{j,p,q}) = (-1)^{j+p+q+1} / K(\lambda). \tag{2.20}$$

Comparing (2.18) with (2.20), we see that the residues of $\widetilde{F}$ and $i\widetilde{G}$ are equal if $j+q$ is odd, and they are opposite if $j+q$ is even. □

We conclude that the function $\widetilde{F}(z) + i\widetilde{G}(z)$ has only $2n$ finite nonzero poles,

$$\pm\left(\ell \operatorname{sn}(iv_{2j-1}, \ell) + (-1)^{j+1} i \operatorname{dn}(iv_{2j-1}, \ell)\right), \quad j = 1, 2, \dots, n.$$

All of these poles are simple. Since $\widetilde{F}(z) + i\widetilde{G}(z)$ has unit modulus on the unit circle, its finite nonzero roots are the reciprocals of these poles.

Now observe that since $\operatorname{sn}(iu, \ell) = i\frac{\operatorname{sn}(u,\ell')}{\operatorname{cn}(u,\ell')}$ and $\operatorname{dn}(iu, \ell) = \frac{\operatorname{dn}(u,\ell')}{\operatorname{cn}(u,\ell')}$ [12, Equation 2.6.12], we have

$$\begin{aligned}
\left(\ell \operatorname{sn}(iv_{2j-1}, \ell) + (-1)^{j+1} i \operatorname{dn}(iv_{2j-1}, \ell)\right)^2 &= \left(\ell \operatorname{sn}(iv_{2j-1}, \ell) + i \operatorname{dn}(iv_{2j-1}, \ell)\right)^{2(-1)^{j+1}} \\
&= \left(\frac{i\ell \operatorname{sn}(v_{2j-1}, \ell') + i \operatorname{dn}(v_{2j-1}, \ell')}{\operatorname{cn}(v_{2j-1}, \ell')}\right)^{2(-1)^{j+1}} \\
&= -\left(\frac{\ell \operatorname{sn}(v_{2j-1}, \ell') + \operatorname{dn}(v_{2j-1}, \ell')}{\operatorname{cn}(v_{2j-1}, \ell')}\right)^{2(-1)^{j+1}} \\
&= -1/a_j, \qquad (2.21)
\end{aligned}$$

where a_j is given by (2.2) (recall that we are still focusing on the case in which $m = 2n+1$ and n is even).

It follows that

$$\widetilde{F}(z) + i\widetilde{G}(z) = e^{i\alpha} z^k \prod_{j=1}^{n} \frac{z^2 + a_j}{1 + a_j z^2} \tag{2.22}$$

for some $\alpha \in \mathbb{R}$ and some $k \in \mathbb{Z}$. We must have $e^{i\alpha} = 1$ since $\widetilde{F}(1) = F(1) > 0$ and $\widetilde{G}(1) = 0$. We must have $k \geq -1$ since $\widetilde{F}(z)$ and $\widetilde{G}(z)$ each have simple poles at $z = 0$. We must have $k \leq 1$ for a similar reason: $\widetilde{F}(z) - i\widetilde{G}(z) = \frac{1}{\widetilde{F}(z) + i\widetilde{G}(z)}$ cannot have a pole of order > 1 at $z = 0$. To conclude, note that at $z = i$, the left-hand side of (2.22) evaluates to i, while the right-hand side evaluates to $i^k(-1)^n = i^k$. The only possibility is $k = 1$. Thus,

$$\widetilde{F}(z) + i\widetilde{G}(z) = z \prod_{j=1}^{n} \frac{z^2 + a_j}{1 + a_j z^2} = \frac{z}{r_n(z^2; \Theta)}, \qquad \text{if } m = 2n+1 \text{ and } n \text{ is even.}$$

In view of (2.5), this completes the proof of Theorem 2.4 for the case in which $m = 2n + 1$ and n is even.

Case 2 ($m = 2n+1$, n odd). The case in which $m = 2n+1$ and n is odd is handled similarly. This time, (2.21) becomes

$$\big(\ell \operatorname{sn}(iv_{2j-1}, \ell) + (-1)^{j+1} i \operatorname{dn}(iv_{2j-1}, \ell)\big)^2 = -a_j,$$

so that (2.22) becomes

$$\widetilde{F}(z) + i\widetilde{G}(z) = e^{i\alpha} z^k \prod_{j=1}^{n} \frac{1 + a_j z^2}{z^2 + a_j}. \tag{2.23}$$

As before, we can argue that $e^{i\alpha} = 1$ and $-1 \le k \le 1$. At $z = i$, the left-hand side evaluates to i, while the right-hand side evaluates to $i^k(-1)^n = -i^k$. We conclude that $k = -1$. That is,

$$\widetilde{F}(z) + i\widetilde{G}(z) = \frac{1}{z} \prod_{j=1}^{n} \frac{1 + a_j z^2}{z^2 + a_j} = \frac{r_n(z^2; \Theta)}{z}, \qquad \text{if } m = 2n + 1 \text{ and } n \text{ is odd.}$$

Case 3 ($m = 2n$). Finally, when $m = 2n$, the identities (2.11-2.12) change to [1, p. 214]

$$\operatorname{sn}\left(\frac{u}{M}, \lambda\right) = \frac{\operatorname{sn}(u, \ell)}{M} \frac{\prod_{k=1}^{n-1} 1 + \operatorname{sn}^2(u, \ell) \frac{\operatorname{cn}^2(v_{2k}, \ell')}{\operatorname{sn}^2(v_{2k}, \ell')}}{\prod_{k=1}^{n} 1 + \operatorname{sn}^2(u, \ell) \frac{\operatorname{cn}^2(v_{2k-1}, \ell')}{\operatorname{sn}^2(v_{2k-1}, \ell')}}, \tag{2.24}$$

$$\operatorname{dn}\left(\frac{u}{M}, \lambda\right) = \prod_{k=1}^{n} \frac{1 - \operatorname{sn}^2(u, \ell) \operatorname{dn}^2(v_{2k-1}, \ell')}{1 + \operatorname{sn}^2(u, \ell) \frac{\operatorname{cn}^2(v_{2k-1}, \ell')}{\operatorname{sn}^2(v_{2k-1}, \ell')}}. \tag{2.25}$$

Note that in contrast to [1, p. 214], we terminated the product in the numerator of (2.24) at $k = n - 1$ rather than $k = n$ since $\operatorname{cn}(v_{2n}, \ell') = 0$ when $m = 2n$. Accordingly, we put

$$\widetilde{F}(z) = \widetilde{F}_{2n}(z; \Theta) = F_{2n}\left(\frac{1}{2}(z + z^{-1}); \ell\right), \tag{2.26}$$

$$\widetilde{G}(z) = \widetilde{G}_{2n}(z; \Theta) = G_{2n}\left(\frac{1}{2}(z + z^{-1}); \ell\right), \tag{2.27}$$

and we observe that $\widetilde{F}(z) + i\widetilde{G}(z)$ is a rational function which is ostensibly of type $(4n, 4n)$. However, $2n$ of the poles $z_{j,p,q}$ coincide and have opposite residues; this time it is those poles $z_{j,p,q}$ for which $j + p$ is even, since the factor $\operatorname{sign}(\operatorname{Im} z_{j,p,q})$ does not appear in the analysis (compare (2.27) with (2.14)). Since $\widetilde{F}$ and $i\widetilde{G}$ have $2n$ coincident poles with opposite residues, $\widetilde{F}(z) + i\widetilde{G}(z)$ is in fact of type $(2n, 2n)$. The poles of $\widetilde{F}(z) + i\widetilde{G}(z)$ are

$$\begin{aligned}(-1)^{j+1} \left(\ell \operatorname{sn}(iv_{2j-1}, \ell) \pm i \operatorname{dn}(iv_{2j-1}, \ell)\right) &= \pm(-1)^{j+1} i b_j^{\pm(-1)^j} \\ &= -(ib_j)^{\pm(-1)^j}, \quad j = 1, 2, \ldots, n,\end{aligned}$$

where b_j is given by (2.4). One checks that the sets $\{-(ib_j)^{(-1)^j}\}_{j=1}^n \cup \{-(ib_j)^{-(-1)^j}\}_{j=1}^n$ and $\{-1/(ib_j)\}_{j=1}^m$ are equal, so $\widetilde{F}(z) + i\widetilde{G}(z)$ must have the form

$$\widetilde{F}(z) + i\widetilde{G}(z) = e^{i\alpha} z^k \prod_{j=1}^m \frac{z - ib_j}{1 + ib_j z}. \tag{2.28}$$

Again, we can argue that $-1 \le k \le 1$, but this time we cannot conclude that $e^{i\alpha} = 1$ by evaluating both sides of (2.28) at $z = 1$. Instead, we evaluate both sides at $z = i$ to obtain $i = e^{i\alpha} i^{k+m}$, and we evaluate both sides at $z = -i$ to obtain $i = e^{i\alpha}(-i)^{k+m}$. We conclude that $k+m$ is even, and since m is too, we have $k = 0$ and $e^{i\alpha} = i^{1-m}$.

This completes the proof of Theorem 2.4.

As a final remark, we note that Theorem 2.4 also holds trivially when $m = 0$ if we adopt the convention that $\lambda := 0$ when $m = 0$.

2.2. **Proof of Theorem 2.1.** Let us now use Theorem 2.4 to prove Theorem 2.1. We first elaborate on the relation between Problems Z5 and Z6. Observe that if $w = z^2$, then $w \in S_\Theta \iff z \in T_\Theta$, and

$$\arg\left(\frac{zp(z^2)/q(z^2)}{\operatorname{sign}(z)}\right) = -\arg\left(\frac{q(w)/p(w)}{\sqrt{w}}\right) = -\arg\left(\frac{q(z^2)/(zp(z^2))}{\operatorname{sign}(z)}\right) \tag{2.29}$$

for any polynomials p and q. In view of (2.10) and Remark 2.2, it follows that $\arg\left(\frac{r(e^{i\theta})}{\sqrt{e^{i\theta}}}\right)$ equioscillates $2n+2$ times on $[-2\Theta, 2\Theta]$, taking values in $[-\arccos\lambda, \arccos\lambda]$. Suppose now that $\widetilde{r}(z)$ is another rational function of type (n, n) satisfying $|\widetilde{r}(z)| = 1$ for $|z| = 1$ and

$$\max_{z \in S_\Theta} \left|\arg\left(\frac{\widetilde{r}(z)}{\sqrt{z}}\right)\right| \le \arccos\lambda.$$

Then the equioscillation of $\arg\left(\frac{r(e^{i\theta})}{\sqrt{e^{i\theta}}}\right)$ implies that on $[-2\Theta, 2\Theta]$,

$$\arg\left(\frac{r(e^{i\theta})}{\sqrt{e^{i\theta}}}\right) - \arg\left(\frac{\widetilde{r}(e^{i\theta})}{\sqrt{e^{i\theta}}}\right) = \arg\left(\frac{r(e^{i\theta})}{\widetilde{r}(e^{i\theta})}\right)$$

has at least $2n + 1$ roots, counted with multiplicity. Hence, the numerator of $r(z) - \widetilde{r}(z)$ has at least $2n + 1$ roots, counted with multiplicity. Since $r(z) - \widetilde{r}(z)$ has type $(2n, 2n)$, it follows that $\widetilde{r} = r$. This shows that Problem Z5 has a unique solution, namely r.

The proof that Problem Z6 has precisely two solutions—$s(z)$ and $s(z)^{-1}$—proceeds similarly. Assume $m > 0$; otherwise the claim is trivial. We see from (2.10) that

$$\max_{z \in T_\Theta} \left|\arg\left(\frac{s(z)}{\operatorname{sign}(z)}\right)\right| = \arccos\lambda,$$

and $\arg\left(\frac{s(e^{i\theta})}{\operatorname{sign}(e^{i\theta})}\right)$ equioscillates $m + 1$ times on $[-\Theta, \Theta]$ and $m + 1$ times on $[\pi - \Theta, \pi + \Theta]$, owing to the fact that $\operatorname{sign}(e^{i\theta}) = 1$ when $\theta \in [-\Theta, \Theta]$, $\operatorname{sign}(e^{i\theta}) = -1$ when $\theta \in [\pi - \Theta, \pi + \Theta]$, and $-s(e^{i\theta}) = s(e^{i(\pi-\theta)})^{-1}$ for all θ. The same is true of $\arg\left(\frac{s(e^{i\theta})^{-1}}{\operatorname{sign}(e^{i\theta})}\right)$ since

$$\arg\left(\frac{s(e^{i\theta})^{-1}}{\operatorname{sign}(e^{i\theta})}\right) = -\arg\left(\frac{s(e^{i\theta})}{\operatorname{sign}(e^{i\theta})}\right)$$

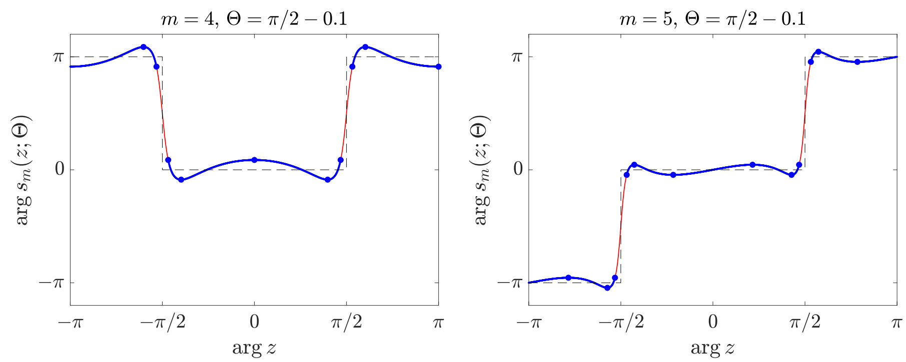

FIGURE 1. Plots of $\arg s_m(z;\Theta)$ with $\Theta = \pi/2 - 0.1$ and $m = 4, 5$. Portions of the graph corresponding to points $z \in T_\Theta$ (respectively, $z \notin T_\Theta$) are colored blue (respectively, red). Extrema of the error on T_Θ are marked with blue dots. The dashed line is $\arg \operatorname{sign} z$. Both the horizontal and vertical axes are to be interpreted modulo 2π. The graphs of $\arg\left(s_m(z;\Theta)^{-1}\right)$ are obtained by reflecting the above graphs across the horizontal axis.

for all θ. Suppose now that $\widetilde{s}(z)$ is another rational function of type (m, m) satisfying $|\widetilde{s}(z)| = 1$ for $|z| = 1$ and

$$\max_{z \in T_\Theta} \left| \arg\left(\frac{\widetilde{s}(z)}{\operatorname{sign} z} \right) \right| \leq \arccos \lambda.$$

Then the same reasoning as above shows that the numerator of $\widetilde{s}(z) - s(z)$ has at least $2m$ roots counted with multiplicity. At least m of them lie in $\{z \in T_\Theta \mid \operatorname{Re} z > 0\}$, and at least m of them lie in $\{z \in T_\Theta \mid \operatorname{Re} z < 0\}$. Likewise, the numerator of $\widetilde{s}(z) - s(z)^{-1}$ has at least $2m$ roots counted with multiplicity, at least m of which lie in $\{z \in T_\Theta \mid \operatorname{Re} z > 0\}$ and at least m of which lie in $\{z \in T_\Theta \mid \operatorname{Re} z < 0\}$. By considering the graphs of $\arg s(z)$ and $\arg(s(z)^{-1})$ (see Figure 1), there must also be at least one point $z \in \{z \in \mathbb{C} \mid |z| = 1,\, z \notin \operatorname{int}(T_\Theta)\}$ where either $\widetilde{s}(z) = s(z)$ or $\widetilde{s}(z) = s(z)^{-1}$. (If all such points happen to be on the boundary of T_Θ, then it is easy to see that there must have been more than $2m$ points in T_Θ (counting multiplicities) where either $\widetilde{s}(z) = s(z)$ or $\widetilde{s}(z) = s(z)^{-1}$ to begin with.) We conclude that either $\widetilde{s} = s$ or $\widetilde{s} = 1/s$. This shows that Problem Z6 has precisely two solutions: s and $1/s$.

3. Properties of the Solutions

In this section, we study the error committed by the functions $r_n(z;\Theta)$ and $s_m(z;\Theta)$ from Theorem 2.1, and we study the behavior of $r_n(z;\Theta)$ and $s_m(z;\Theta)$ under composition.

3.1. **Error.** To study the error, we appeal to well-known properties of the function $F_m(x;\ell)$ defined in (2.8). As we noted earlier, $\frac{2}{1+\lambda}F_m(x;\ell)$ is the solution to Problem Z4 on $[-1, -\ell] \cup [\ell, 1]$.

The number $\frac{1-\lambda}{1+\lambda} = \max_{x \in [-1,-\ell] \cup [\ell,1]} \left| \frac{2}{1+\lambda} F_m(x;\ell) - \operatorname{sign}(x) \right|$ is well-studied; it satisfies [2, p. 9]

$$\frac{1-\lambda}{1+\lambda} = \frac{2\sqrt{Z_m}}{1+Z_m}, \tag{3.1}$$

where $Z_m = Z_m([-1,-\ell],[\ell,1])$ denotes the *Zolotarev number* of the sets $[-1,-\ell]$ and $[\ell,1]$:

$$Z_m(E,F) = \inf_{r \in \mathcal{R}_{m,m}} \frac{\sup_{z \in E} |r(z)|}{\inf_{z \in F} |r(z)|}. \tag{3.2}$$

An explicit formula for Z_m ($m \geq 1$) is [2, Theorem 3.1]

$$Z_m = 4\rho^{-2m} \prod_{j=1}^{\infty} \frac{(1+\rho^{-8jm})^4}{(1+\rho^{4m}\rho^{-8jm})^4} \leq 4\rho^{-2m},$$

where

$$\rho = \exp\left(\frac{\pi K(\ell)}{K(\ell')}\right) = \exp\left(\frac{\pi K(\cos\Theta)}{K(\sin\Theta)}\right).$$

Note that the bound $Z_m \leq 4\rho^{-2m}$ also obviously holds for $m = 0$. Solving for λ in (3.1), we find that

$$\max_{z \in T_\Theta} \left| \arg\left(\frac{s_m(z;\Theta)}{\operatorname{sign}(z)}\right) \right| = \arccos\lambda = \arccos\left(\left(\frac{1-\sqrt{Z_m}}{1+\sqrt{Z_m}}\right)^2\right). \tag{3.3}$$

We derive upper bounds for this quantity below.

Lemma 3.1. *For every $x \geq 0$,*

$$\arccos\left(\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^2\right) \leq 2\sqrt{2}x^{1/4}.$$

Proof. Let $f(x) = \arccos\left(\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^2\right)$ and $g(x) = 2\sqrt{2}x^{1/4}$. Since $f(0) = g(0) = 0$ and

$$f'(x) = \frac{1-\sqrt{x}}{\sqrt{2}x^{3/4}(1+\sqrt{x})\sqrt{1+x}} < \frac{1}{\sqrt{2}x^{3/4}} = g'(x), \quad x > 0,$$

we have $f(x) = \int_0^x f'(t)\,dt \leq \int_0^x g'(t)\,dt = g(x)$ for every $x \geq 0$. □

Theorem 3.2. *Let $\Theta \in (0,\pi/2)$ and $m,n \in \mathbb{N}_0$. We have*

$$\max_{z \in T_\Theta} \left| \arg\left(\frac{s_m(z;\Theta)}{\operatorname{sign}(z)}\right) \right| \leq 4\rho^{-m/2} \leq 4\left[\exp\left(\frac{\pi^2}{4\log(4\sec\Theta)}\right)\right]^{-m} \tag{3.4}$$

and

$$\max_{z \in S_\Theta} \left| \arg\left(\frac{r_n(z;\Theta)}{\sqrt{z}}\right) \right| \leq 4\rho^{-(n+1/2)} \leq 4\left[\exp\left(\frac{\pi^2}{2\log(4\sec\Theta)}\right)\right]^{-(n+1/2)}. \tag{3.5}$$

Proof. Using Lemma 3.1 and the inequality [2, p. 8]

$$\frac{\pi}{2}K(\sqrt{1-x^2})/K(x) \leq \log 4/x, \quad 0 < x < 1,$$

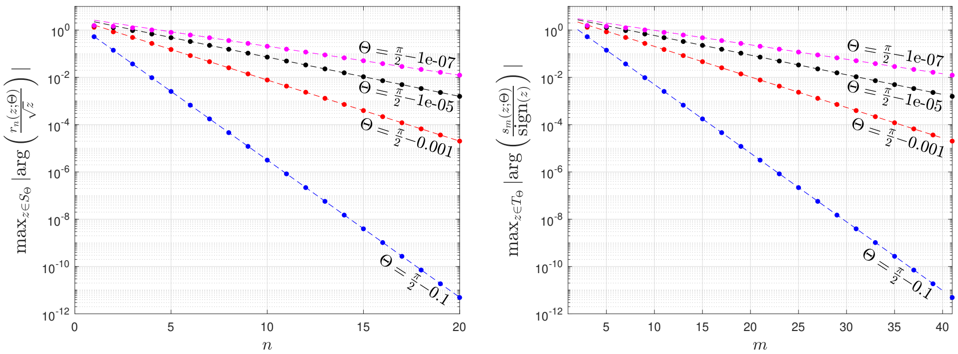

FIGURE 2. The errors (dots) and their bounds in Theorem 3.2 (dashed lines, the rightmost bounds in (3.4), (3.5) are shown) for Z5 (left) and Z6 (right).

we compute

$$\max_{z\in T_\Theta}\left|\arg\left(\frac{s_m(z;\Theta)}{\operatorname{sign}(z)}\right)\right| \le 2\sqrt{2}Z_m^{1/4} \le 2\sqrt{2}\left(4\rho^{-2m}\right)^{1/4} = 4\rho^{-m/2}$$
$$\le 4\left[\exp\left(\frac{\pi^2}{2\log(4\sec\Theta)}\right)\right]^{-m/2}.$$

The bound (3.5) follows from Remark 2.2 and (2.29), which imply

$$\max_{z\in S_\Theta}\left|\arg\left(\frac{r_n(z;\Theta)}{\sqrt{z}}\right)\right| = \max_{z\in T_\Theta}\left|\arg\left(\frac{s_{2n+1}(z;\Theta)}{\operatorname{sign}(z)}\right)\right|.$$

□

Theorem 3.2 is illustrated in Figure 2, which shows the bounds are very tight.
Figure 3 plots the absolute errors $|r_n(z;\Theta) - \sqrt{z}|$ and $|s_m(z;\Theta) - \operatorname{sign}(z)|$ for $z \in \mathbb{C}$.

3.2. **Composition.** Next, we show that when two solutions of Problem Z6 are composed with one another, the resulting function is a solution of Problem Z6 of higher degree.

Theorem 3.3. *Let* $\Theta \in (0, \pi/2)$, $m, \widetilde{m} \in \mathbb{N}_0$, *and* $\widetilde{\Theta} = \left|\arg(s_m(e^{i\Theta};\Theta))\right|$. *Then*

$$s_{\widetilde{m}}(s_m(z;\Theta);\widetilde{\Theta}) = s_{\widetilde{m}m}(z;\Theta).$$

Proof. This is essentially a consequence of the identities

$$f_{\widetilde{m}\widetilde{\nu}} \circ f_{\widetilde{\nu}}^{-1} \circ f_{m\nu} \circ f_\nu^{-1} = f_{\widetilde{m}m\nu} \circ f_\nu^{-1}, \tag{3.6}$$

$$g_{\widetilde{m}\widetilde{\nu}} \circ f_{\widetilde{\nu}}^{-1} \circ f_{m\nu} \circ f_\nu^{-1} = \pm g_{\widetilde{m}m\nu} \circ f_\nu^{-1}, \tag{3.7}$$

which hold on $[-1, 1]$ whenever

$$\widetilde{\nu} = m\nu. \tag{3.8}$$

(The $\pm$ sign in (3.7) is $+$ at x if $g_{m\nu}(f_\nu^{-1}(x))^{\widetilde{m}}$ is positive and $-$ at x if $g_{m\nu}(f_\nu^{-1}(x))^{\widetilde{m}}$ is negative, owing to the branch cut structure of sn^{-1}.)

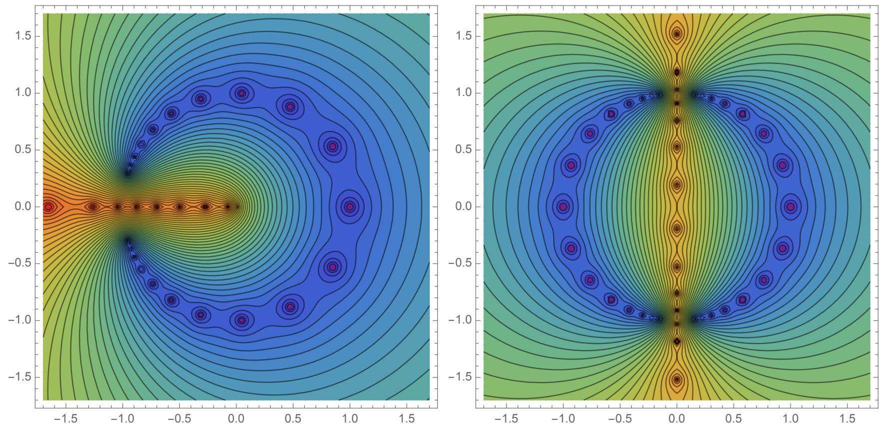

FIGURE 3. Contours of the error $|r_n(z;\Theta) - \sqrt{z}|$ (left) and $|s_m(z;\Theta) - \operatorname{sign}(z)|$ (right) in the complex plane with $n = 11$, $m = 17$, and $\Theta = \frac{\pi}{2} - 0.15$. The extrema on the unit circle are the zeros of the error, and the extrema on the coordinate axes are poles of the approximants.

To flesh out the details, note that (3.8) holds for $\nu = 1/\mu(\ell)$ and $\widetilde{\nu} = 1/\mu(\widetilde{\ell})$ if and only if

$$\frac{K(\ell)}{K(\ell')} = \frac{K(\widetilde{\ell})}{mK(\widetilde{\ell}')}.$$

Comparing with (2.6), we see that this happens precisely when $\widetilde{\ell} = \lambda = F_m(\ell;\ell)$. In turn, this holds if and only if $\ell = \cos\Theta$ and $\widetilde{\ell} = \cos\widetilde{\Theta}$ with $\widetilde{\Theta} = \left|\arg(s_m(e^{i\Theta};\Theta))\right|$.

Let us now compute $s_{\widetilde{m}}(s_m(z;\Theta);\widetilde{\Theta})$ under the assumption that $\widetilde{\Theta} = \left|\arg(s_m(e^{i\Theta};\Theta))\right|$. Since

$$s_m(z;\Theta) = \widetilde{F}_m(z;\Theta) + i\widetilde{G}_m(z;\Theta) = f_{m\nu}(f_\nu^{-1}(x)) + i(\operatorname{sign}\operatorname{Im} z)^m g_{m\nu}(f_\nu^{-1}(x))$$

and $s_m(z;\Theta)^{-1} = \widetilde{F}_m(z;\Theta) - i\widetilde{G}_m(z;\Theta)$, we have

$$\frac{1}{2}(s_m(z;\Theta) + s_m(z;\Theta)^{-1}) = \widetilde{F}_m(z;\Theta) = f_{m\nu}(f_\nu^{-1}(x)),$$

where

$$x = \frac{1}{2}(z + z^{-1}), \quad \nu = \frac{1}{\mu(\ell)}, \quad \ell = \cos\Theta.$$

Thus, denoting

$$\begin{aligned}
&\widetilde{\nu} = 1/\mu(\cos\widetilde{\Theta}) = m\nu, \\
&\sigma = (\operatorname{sign}\operatorname{Im} s_m(z;\Theta))^{\widetilde{m}} = (\operatorname{sign}\operatorname{Im} z)^{\widetilde{m}m}\operatorname{sign}(g_{m\nu}(f_\nu^{-1}(x)))^{\widetilde{m}}, \\
&\tau = \operatorname{sign}(g_{m\nu}(f_\nu^{-1}(x)))^{\widetilde{m}},
\end{aligned}$$

we find

$$\begin{aligned}
s_{\widetilde{m}}(s_m(z;\Theta);\widetilde{\Theta}) &= f_{\widetilde{m}\widetilde{\nu}}(f_{\widetilde{\nu}}^{-1}(f_{m\nu}(f_\nu^{-1}(x)))) + i\sigma g_{\widetilde{m}\widetilde{\nu}}(f_{\widetilde{\nu}}^{-1}(f_{m\nu}(f_\nu^{-1}(x)))) \\
&= f_{\widetilde{m}m\nu}(f_\nu^{-1}(x)) + i\sigma\tau g_{\widetilde{m}m\nu}(f_\nu^{-1}(x)) \\
&= s_{\widetilde{m}m}(z;\Theta),
\end{aligned}$$

where the last line follows from the fact that $\sigma\tau = (\operatorname{sign}\operatorname{Im} z)^{\widetilde{m}m}$. □

We illustrate Theorem 3.3 in Figure 4.

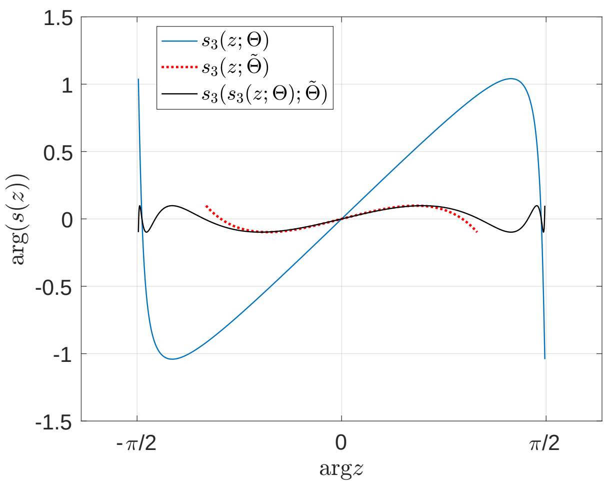

FIGURE 4. Illustration of Theorem 3.3 for $m = \widetilde{m} = 3$, $\Theta = \pi/2 - 0.01$: $s_3(z;\Theta)$, $s_3(z;\widetilde{\Theta})$ and $s_3(s_3(z;\Theta);\widetilde{\Theta}) = s_9(z;\Theta)$. Only $[-\Theta,\Theta]$ is shown; by symmetry the plots look the same on $[\pi - \Theta, \pi + \Theta]$. Composing low-degree solutions results in a high-degree solution.

Remark 3.4. Theorem 3.3 can also be proved by counting equioscillation points. As θ runs from $-\Theta$ to Θ, the number $\widetilde{\theta} := \arg\left(s_m(e^{i\theta};\Theta)\right)$ equioscillates $m+1$ times, taking values in $[-\widetilde{\Theta},\widetilde{\Theta}]$ and achieving its extrema at the endpoints. Each time $\widetilde{\theta}$ runs from $\pm\widetilde{\Theta}$ to $\mp\widetilde{\Theta}$, the number

$$\widehat{\theta} := \arg\left(s_{\widetilde{m}}(e^{i\widetilde{\theta}};\widetilde{\Theta})\right) = \arg\left(s_{\widetilde{m}}(s_m(e^{i\theta};\Theta);\widetilde{\Theta})\right)$$

equioscillates $\widetilde{m}+1$ times, achieving its extrema at the endpoints. By counting extrema, we see that as θ runs from $-\Theta$ to Θ, $\widehat{\theta}$ equioscillates $\widetilde{m}m+1$ times. Since $s_{\widetilde{m}}(s_m(z;\Theta))$ is a rational function of degree $\widetilde{m}m$, we can argue as we did in the proof of Theorem 2.1 that $s_{\widetilde{m}}(s_m(z;\Theta))$ must be a solution of Problem Z6. Hence, $s_{\widetilde{m}}(s_m(z;\Theta);\widetilde{\Theta}) = s_{\widetilde{m}m}(z;\Theta)^\sigma$ for some $\sigma \in \{-1,1\}$. Evaluating both sides of this equation at $z = i$ yields $\sigma = 1$, so $s_{\widetilde{m}}(s_m(z;\Theta);\widetilde{\Theta}) = s_{\widetilde{m}m}(z;\Theta)$.

Remark 3.5. The identity (3.6) shows that Zolotarev's (scaled) minimax approximant $F_m(x;\ell)$ of $\operatorname{sign}(x)$ on $[-1,-\ell]\cup[\ell,1]$ satisfies

$$F_{\widetilde{m}}(F_m(x;\ell);\widetilde{\ell}) = F_{\widetilde{m}m}(x;\ell) \tag{3.9}$$

whenever $\widetilde{\ell} = F_m(\ell;\ell)$. This composition law has been studied in, for example, [3, 4, 16].

Remark 3.6. It is not hard to check that the function $\widetilde{s}_{2n+1}(z;\Theta) := s_{2n+1}(z;\Theta)^{(-1)^n}$ also behaves nicely under composition: If $\widetilde{\Theta} = \left|\arg(\widetilde{s}_{2n+1}(e^{i\Theta};\Theta))\right|$, then

$$\widetilde{s}_{2\widetilde{n}+1}(\widetilde{s}_{2n+1}(z;\Theta);\widetilde{\Theta}) = \widetilde{s}_{(2\widetilde{n}+1)(2n+1)}(z;\Theta).$$

Since

$$\widetilde{s}_{2n+1}(z;\Theta) = \frac{z}{r_n(z^2;\Theta)},$$

we obtain from Theorem 3.3 an analogous composition law for solutions of Problem Z5.

Corollary 3.7. *Let* $\Theta \in (0,\pi/2)$, $\widetilde{n}, n \in \mathbb{N}_0$, *and* $\widetilde{\Theta} = \left|\arg(s_{2n+1}(e^{i\Theta};\Theta))\right| = \left|\arg\left(e^{i\Theta}/r_n(e^{2i\Theta};\Theta)\right)\right|$. *Then*

$$r_n(z;\Theta) r_{\widetilde{n}}\left(\frac{z}{r_n(z;\Theta)^2};\widetilde{\Theta}\right) = r_{2\widetilde{n}n+\widetilde{n}+n}(z;\Theta). \tag{3.10}$$

Remark 3.8. This behavior closely parallels the behavior of rational minimax approximants of $\sqrt{x}$ on positive real intervals; see [6, 7].

Proof. We have

$$\begin{aligned}
\frac{\sqrt{z}}{r_{2\widetilde{n}n+\widetilde{n}+n}(z;\Theta)} &= \widetilde{s}_{4\widetilde{n}n+2\widetilde{n}+2n+1}(\sqrt{z};\Theta) \\
&= \widetilde{s}_{2\widetilde{n}+1}(\widetilde{s}_{2n+1}(\sqrt{z};\Theta);\widetilde{\Theta}) \\
&= \widetilde{s}_{2\widetilde{n}+1}\left(\frac{\sqrt{z}}{r_n(z;\Theta)};\widetilde{\Theta}\right) \\
&= \frac{\sqrt{z}/r_n(z;\Theta)}{r_{\widetilde{n}}\left(z/r_n(z;\Theta)^2;\widetilde{\Theta}\right)}.
\end{aligned}$$

Rearranging this yields (3.10). □

3.3. **Connections with other functions.** We conclude this section by drawing a few connections between the solutions to Problems Z5-Z6 and other well-studied functions.

Finite Blaschke products. Ng and Tsang [17,18] study a finite Blaschke product that behaves nicely under composition and solves the extremal problem (3.2) for $Z_m(E,F)$ with $E = [-\sqrt{\ell},\sqrt{\ell}]$ and $F = (-\infty, -\frac{1}{\sqrt{\ell}}] \cup [\frac{1}{\sqrt{\ell}}, \infty)$. The function is

$$h_m(z;\ell) = \prod_{j=1}^m \frac{z-c_j}{1-c_j z},$$

where

$$c_j = \frac{\sqrt{\ell}\,\mathrm{cn}\left(\frac{2j-1}{m}K(\ell),\ell\right)}{\mathrm{dn}\left(\frac{2j-1}{m}K(\ell),\ell\right)}.$$

They show that if $\widetilde{\ell} = Z_m\left([-\sqrt{\ell},\sqrt{\ell}], (-\infty,-\frac{1}{\sqrt{\ell}}] \cup [\frac{1}{\sqrt{\ell}},\infty)\right)$, then [17, Proposition 2]

$$h_{\widetilde{m}}(h_m(z;\ell);\widetilde{\ell}) = h_{\widetilde{m}m}(z;\ell)$$

for any positive integers $\widetilde{m}$ and m, and [18, Proposition 4.1(b)]

$$\left(\frac{1-\widetilde{\ell}}{1+\widetilde{\ell}}\right)\frac{h_m(z;\ell)-1}{h_m(z;\ell)+1} = \frac{2}{1+F_m(\kappa;\kappa)} F_m(x;\kappa),$$

where

$$x = \kappa\left(\frac{1+\sqrt{\ell}}{1-\sqrt{\ell}}\right)\frac{z-1}{z+1}, \quad \kappa = \left(\frac{1-\sqrt{\ell}}{1+\sqrt{\ell}}\right)^2.$$

Our function s_m is thus related to theirs via

$$\left(\frac{1-\widetilde{\ell}}{1+\widetilde{\ell}}\right)\frac{h_m(z;\ell)-1}{h_m(z;\ell)+1} = \frac{1}{1+F_m(\kappa;\kappa)}\left(s_m(w;\Phi)+s_m(w;\Phi)^{-1}\right),$$

where

$$\frac{1}{2}(w+w^{-1}) = \left(\frac{1-\sqrt{\ell}}{1+\sqrt{\ell}}\right)\frac{z-1}{z+1}, \quad \cos\Phi = \left(\frac{1-\sqrt{\ell}}{1+\sqrt{\ell}}\right)^2 = \kappa.$$

Padé approximants. In the limit as $\Theta \downarrow 0$, the solution to Problem Z5 reduces to a Padé approximant of $\sqrt{z}$. More precisely, let $p_n(z)$ denote the type-(n,n) Padé approximant to $\sqrt{z}$ at $z=1$. An explicit formula for $p_n(z)$ is [6, p. 707]

$$p_n(z) = \sqrt{z}\frac{(1+\sqrt{z})^{2n+1}+(1-\sqrt{z})^{2n+1}}{(1+\sqrt{z})^{2n+1}-(1-\sqrt{z})^{2n+1}}.$$

We say that a parametrized family of rational functions $r(z;\Theta)$ converges coefficientwise to $p_n(z)$ as $\Theta \downarrow 0$ if the coefficients in the numerator and denominator of $r(z;\Theta)$, appropriately normalized, converge to those of $p_n(z)$ as $\Theta \downarrow 0$.

Proposition 3.9. *Let $n \in \mathbb{N}_0$. As $\Theta \downarrow 0$, $r_n(z;\Theta)$ converges coefficientwise to $p_n(z)$.*

Proof. Since $|r_n(z;\Theta)| = |p_n(z)| = 1$ for all z with $|z|=1$, it suffices to show that the poles of $r_n(z;\Theta)$ approach the poles of $p_n(z)$ as $\Theta \downarrow 0$. It is easy to check that the poles of $p_n(z)$ are $\left\{-\tan^2\left(\frac{j\pi}{2n+1}\right)\right\}_{j=1}^n$. On the other hand, the poles of $r_n(z;\Theta)$ are $\{-a_j\}_{j=1}^n$. Since $\lim_{\Theta\downarrow 0} K(\Theta) = K(0) = \pi/2$, $\lim_{\ell'\downarrow 0}\operatorname{sn}(z,\ell') = \operatorname{sn}(z,0) = \sin z$, $\lim_{\ell'\downarrow 0}\operatorname{cn}(z;\ell') = \operatorname{cn}(z,0) = \cos z$, and $\lim_{\ell'\downarrow 0}\operatorname{dn}(z,\ell') = \operatorname{dn}(z,0) = 1$ [5, Table 22.5.3], we have

$$\lim_{\Theta\downarrow 0} a_j = \left(\frac{\sin(2j-1)\omega+1}{\cos(2j-1)\omega}\right)^{2(-1)^{j+n}},$$

where $\omega = \pi/(4n+2)$. Using the identities $\frac{\sin\theta+1}{\cos\theta} = \cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$ and $\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$, this can be simplified to

$$\begin{aligned}\lim_{\Theta\downarrow 0} a_j &= (\cot(n-j+1)\omega)^{2(-1)^{j+n}} \\ &= \begin{cases}\tan^2(n-j+1)\omega, & \text{if } j+n \text{ is odd,} \\ \tan^2(n+j)\omega, & \text{if } j+n \text{ is even.}\end{cases}\end{aligned}$$

This shows that $\{\lim_{\Theta\downarrow 0} a_j\}_{j=1}^n$ contains the squared tangent of every even multiple of ω. Hence,

$$\{-\lim_{\Theta\downarrow 0} a_j\}_{j=1}^n = \left\{-\tan^2\left(\frac{j\pi}{2n+1}\right)\right\}_{j=1}^n.$$

□

Chebyshev polynomials. It is interesting to note the similarity between the results in this paper and the defining property of the Chebyshev polynomials of the first kind $T_n(x)$:

$$\mathrm{Re}(z^n) = T_n(\mathrm{Re}\, z), \quad \text{if } |z| = 1.$$

In fact, we can write Theorem 2.4 in a more suggestive way by denoting

$$\begin{aligned} \mathfrak{F}_m \colon [-1,1]\times(0,1) &\to [-1,1]\times(0,1), \\ (x,\ell) &\mapsto \big(F_m(x;\ell), F_m(\ell;\ell)\big), \end{aligned}$$

$$\begin{aligned} \mathfrak{s}_m \colon \mathbb{S}\times\mathbb{S}_+ &\to \mathbb{S}\times\mathbb{S}_+, \\ (z,\zeta) &\mapsto \big(s_m(z;|\arg\zeta|), s_m(\zeta;|\arg\zeta|)\big), \end{aligned}$$

and

$$\begin{aligned} J\colon \mathbb{S}\times\mathbb{S}_+ &\to [-1,1]\times(0,1), \\ (z,\zeta) &\mapsto (\mathrm{Re}\, z, \mathrm{Re}\,\zeta), \end{aligned}$$

where $\mathbb{S} = \{z\in\mathbb{C} \mid |z| = 1\}$ and $\mathbb{S}_+ = \{z\in\mathbb{S} \mid 0 < \mathrm{Re}\, z < 1\}$. With this notation, Theorem 2.4 says that

$$\mathfrak{F}_m \circ J = J \circ \mathfrak{s}_m, \tag{3.11}$$

and Theorem 3.3 says that

$$\mathfrak{s}_{\widetilde{m}} \circ \mathfrak{s}_m = \mathfrak{s}_{\widetilde{m}m}. \tag{3.12}$$

By combining (3.11) with (3.12), we deduce

$$\mathfrak{F}_{\widetilde{m}} \circ \mathfrak{F}_m = \mathfrak{F}_{\widetilde{m}m}, \tag{3.13}$$

which is a restatement of (3.9). The identities (3.11-3.13) mimic the following identities involving the monomials $t_n(z) = z^n$ and the Chebyshev polynomials $T_n(x)$:

$$T_n \circ \mathrm{Re}\,|_{\mathbb{S}} = \mathrm{Re}\, \circ t_n|_{\mathbb{S}}, \quad t_m \circ t_n = t_{mn}, \quad T_m \circ T_n = T_{mn}.$$

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