# Derivatives of the Sine and Cosine Functions
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As always, we need to assume that an angle θ is measured in radians. Place this angle at the origin so that the positive x-axis is one of the two rays forming the angle. Now draw a circle around the origin with radius 1. The point P where the second ray of the angle intersects this circle will have coordinates $(\cos\theta,\, \sin\theta)$. Furthermore, the arc of the circle connecting this point with the x-axis will have length θ. (See picture below. For convenience, we assume that θ is positive and smaller than $\pi/2$. The conscientious read will draw pictures to make sure that the reasoning is still valid in other cases.)

Thus if θ increases at a rate of 1 radian/second, then the point P is moving at a speed of 1 unit/second. Since $\sin\theta$ is the y-coordinate of this point, the derivative of $\sin\theta$ will be the rate at which the height of P is changing. This will be dependent on the angle which the direction of P's motion (i. e. the tangent to the circle) makes to the vertical.
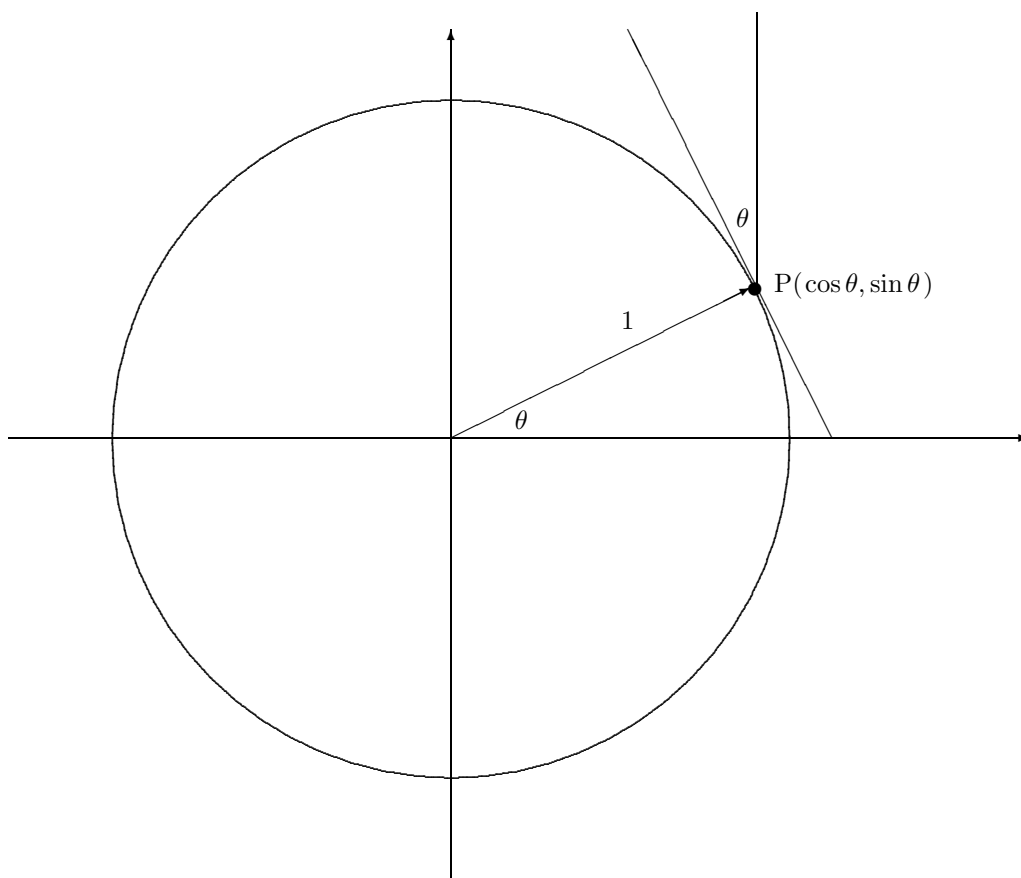
Since the tangent to the circle is perpendicular to the radius, and of course the vertical line shown is perpendicular to the x-axis, easy geometry shows that the angle between the tangent and the vertical is θ. So as P moves along the circle, P is moving with a speed of 1 unit per second in a direction with an angle of θ to the vertical. It should then be clear that the height of P (i. e. its y-coordinate) increases at a rate of $\cos\theta$.

Another way of saying this is that the motion of the point P can be represented by a vector $\mathbf{v}$ with magnitude 1 in the direction of the tangent to the circle. The derivative of $\sin\theta$, which is the rate at which the y-coordinate of P is changing, is then the vertical component of this velocity vector. But since the angle between and the vertical is θ, the magnitude of its vertical component is $|\cos\theta|$. Furthermore, this vertical component will be directed downward when P is in to the left of the y-axis, which is precisely when $\cos\theta$ is negative.
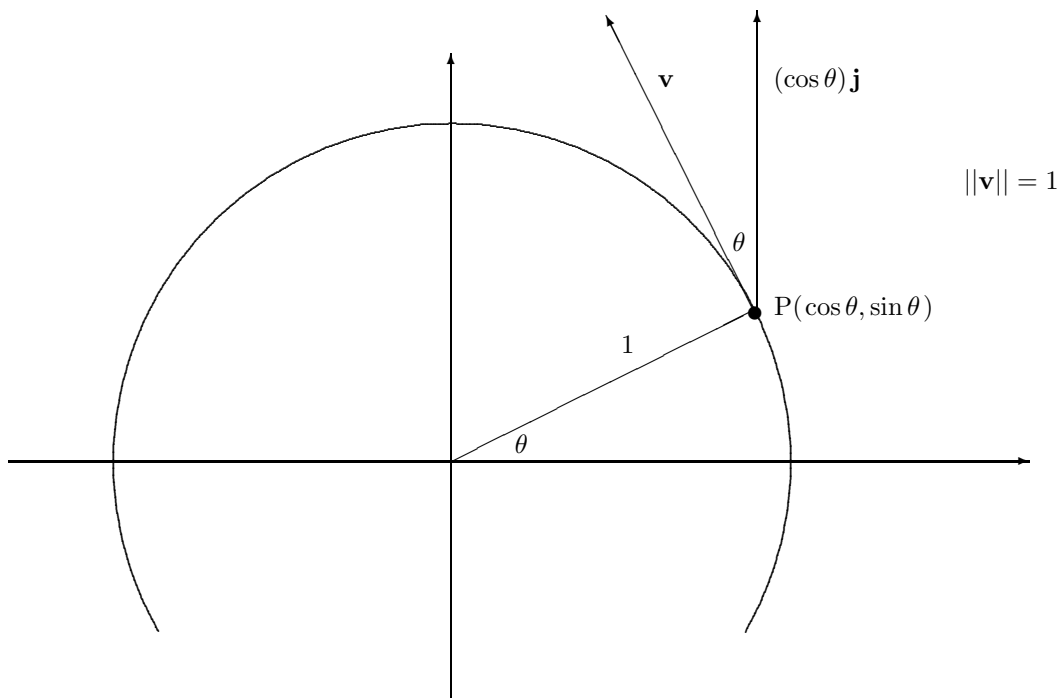
Analogous reasoning (noticing that the rate at which the x-coordinate of P is changing corresponds to the horizontal component of $\mathbf{v}$) easily shows that the derivative of $\cos\theta$ is $-\sin\theta$.

One can easily now obtain the derivative formulas for $\tan\theta$, $\sec\theta$, etc. using the quotient rule. One can also derive the derivative formulas for $\tan\theta$ and $\cot\theta$ geometrically, although perhaps understanding the explanation is more trouble than it's worth.

Anyway, for what it's worth, consider the following picture:
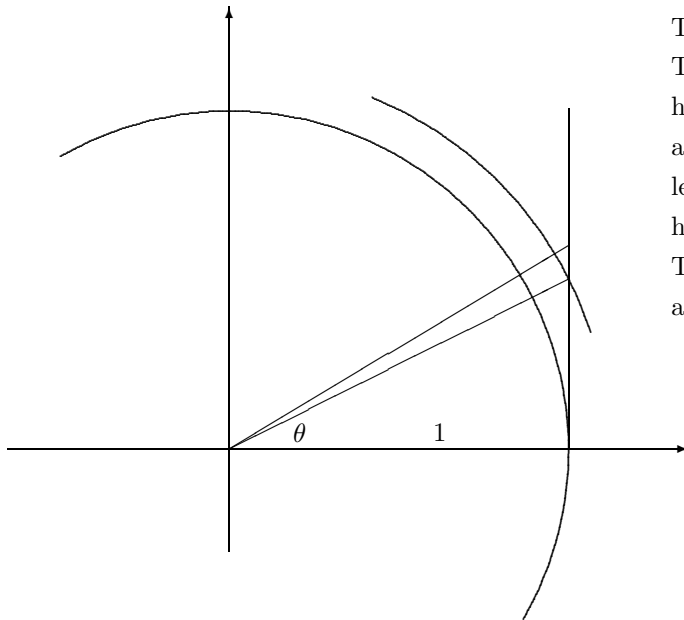
The inner circle has radius 1. The lower triangle has a horizontal side with length 1, a vertical side with length $\tan\theta$, and a hypotenuse of length $\sec\theta$. This gives the lower triangle an area of $\frac{1}{2}\tan\theta$.

If θ is now increased by an amount $\Delta\theta$, this creates within the outside circle with radius $\sec\theta$ a small sector with central angle $\Delta\theta$. Since the area of the outside circle is $\pi\sec^2\theta$, the area of this sector will be $\frac{1}{2}\sec^2\theta\,\Delta\theta$. For $\Delta\theta$ small, this is a very good approximation for the difference in area between the big triangle and the smaller one.

Thus we have

$$\Delta(\text{triangle area}) = \Delta(\tfrac{1}{2}\tan\theta) \approx \tfrac{1}{2}\sec^2\theta\,\Delta\theta.$$

(In fact, assuming that θ is in the first quadrant and $\Delta\theta$ is positive, we have

$$\tfrac{1}{2}\sec^2\theta\,\Delta\theta \le \Delta(\text{Area}) \le \tfrac{1}{2}\sec^2(\theta+\Delta\theta)\,\Delta\theta\,.\,)$$

Dividing through by $\Delta\theta$ and passing to the limit, we get that

$$\frac{d}{dx}(\tan\theta) = \sec^2\theta\,.$$