Green's Theorem

E. L. Lady

February 14, 2000

One of the things that makes Green's Theorem

$$\oint_{\mathcal{C}} Pdx + Qdy = \iint_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx \, dy$$

[where $\mathcal{C}$ is a simple closed curve and P and Q are functions of x and y which have continuous partial derivatives in the region enclosed by $\mathcal{C}$] look more intimidating than it is is that it's actually two theorems written as one:

$$\oint_{\mathcal{C}} Q \, dy = \iint_{\Omega} \frac{\partial Q}{\partial x} \, dx \, dy \,, \qquad \oint_{\mathcal{C}} P \, dx = - \iint_{\Omega} \frac{\partial P}{\partial y} \, dx \, dy.$$

Another thing that makes the basic idea not as clear as it should be is that there's no notation for what ought to be called a **partial integral**, in analogy to a partial derivative. In other words, when one writes $\int f(x,y) \, dx$, there are two ways of interpreting this. In the "partial integral," y is treated as a parameter which, for the purposes of the integration, functions as if it were a constant. This "partial integral" is what one uses in the process of evaluating a double integral $\iint_{\Omega} f(x,y) \, dx \, dy$. On the other hand, when doing a line integral $\int_{\mathcal{C}} P(x,y) \, dx$, one does **not** use a partial integral: here, y is not independent of x but instead represents a function of x which must be taken into account when doing the integral. (In practice, it is often better to think of both x and y as functions of some third variable t, but the point is that y is not held constant during the integration.)

Now if we let the integral sign for the moment represent a "partial" integral, then this is the opposite of the partial derivative, so that

$$\int \frac{\partial P(x,y)}{\partial y} \, dy = P(x,y) + C(x).$$

Here $C(x)$ is the "constant" of integration, except that since we are only integrating y, it may actually be a function of x. When doing a double integral, we can ignore this "constant" $C(x)$ (EXPLAIN!) and we get

$$\iint \frac{\partial P(x,y)}{\partial y}\, dx\, dy = \int P(x,y)\, dx,$$

which seems to be essentially Green's Theorem.

Now this seems more or less plausible, but if a student is as skeptical as s/he ought to be, this "proof" of Green's Theorem will bother him [her] a little bit. There are in fact several things that seem a little puzzling. It takes a while to notice all of them, but the puzzlements are as follows:

(1) Why does the actual Green's Theorem have a minus sign where the formula we just derived does not? (This is the most obvious signal that the "proof" given above is sliding over some subtleties.)

(2) Why is the single integral in Green's Theorem a line integral?

(3) When we get P by integrating $\dfrac{\partial P}{\partial y}$ with respect to y, why does the answer still have a y in it?

If the student is not confused at this point, s/he should stop now and take the time to think about it until s/he becomes confused. Confusion is an essential part of the process of learning to understand what's really going on.

As an example, let's see how this works out for $P(x,y) = y^3 \cos 5x$ if we let Ω be the region bounded by the circle $\mathcal{C}$ around the origin with radius 2. The top boundary of Ω is the graph of the function $y = \sqrt{4 - x^2}$ and the bottom is the curve $y = -\sqrt{4 - x^2}$, where in both cases x ranges from -2 to 2. Then $\dfrac{\partial P}{\partial y} = 3y^2 \cos 5x$ and

$$\iint_\Omega \frac{\partial P}{\partial y}\, dy\, dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{\partial P}{\partial y}\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 3y^2 \cos 5x\, dy\, dx$$

$$= \int_{-2}^{2} y^3 \cos 5x \left.\right|_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\, dx$$

$$= 2 \int_{-2}^{2} (4 - x^2)^{3/2} \cos 5x\, dx\,.$$

Notice how y disappears after we do the inside integral. The plausible formula which we wrote in the abstract as

$$\iint_\Omega \frac{\partial P}{\partial y}\, dy\, dx = \int P\, dx$$

turns out to be a little misleading once we fill in the details. In this example it should actually look like

$$\iint_\Omega \frac{\partial P}{\partial y}\, dy\, dx = \int_{-2}^{2} P(x,y)\Big|_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\, dx\,.$$

On the other hand, Green's Theorem asserts that this should equal

$$-\oint_{\mathcal{C}} P(x,y)\, dx = -\oint_{\mathcal{C}} y^3 \cos 5x\, dx\,,$$

where $\mathcal{C}$ is the boundary of Ω, i.e the circle with radius 2 centered at the origin.

The two formulas certainly have some ressemblance, but it is far from obvious that they are the same.

LINE INTEGRALS

In order to understand Green's Theorem, you have to understand line integrals in the most down-to-earth way possible. In fact, it's going to be important to really run the explanation into the ground.

We start with a curve $\mathcal{C}$ and a vector parametrization $\mathbf{r}(t)$ for this curve. In general, we denote a point on the curve by (x,y), where $x = x(t)$, $y = y(t)$. Thus $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$. We also have a vector field $\mathbf{F}(x,y)$ defined in the whole plane, or at least enough of the plane to include the curve. (For Green's Theorem, it is essential that $\mathbf{F}(x,y)$ be defined and differentiable on the region inside the curve as well as on the curve itself.) This is why $\mathbf{F}$ is written as a function of the point (x,y) and not a function of t. It is standard to denote the components of the vector $\mathbf{F}(x,y)$ by P and Q:

$$\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}.$$

Now we want to define the concept of the line integral of $\mathbf{F}$ along $\mathcal{C}$. The most conceptual notation for this is

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr}.$$

("Conceptual" in this case means that the formula does not refer to the coordinates x and y. Thus the formula does not change when we make a change of coordinates. In

general, the more "conceptual" a formula is, the less practical it is for doing an actual calculation.)

Now I want to put aside all considerations of what this formula really means (in other words, all "conceptual" considerations) and simply try to understand this line integral in terms of a simple integral by substitution.

We have the following formulas:

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$$
$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$
$$\mathbf{dr}(t) = \mathbf{r}'(t)\,dt$$
$$= (\,x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}\,)\,dt$$
$$\mathbf{F}(x, y) \cdot \mathbf{dr} = [P(x, y)x'(t) + Q(x, y)y'(t)]\,dt.$$

We can put this all together, and also write

$$dx = x'(t)\,dt\,, \qquad\qquad dy = y'(t)\,dt$$

and thus get

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr} = \int_{t_0}^{t_1} [P(t)\,x'(t) + Q(t)\,y'(t)]\,dt$$
$$= \int_{\mathcal{C}} P\,dx + Q\,dy,$$

where in the first line we have used a notation common in physics but generally avoided by mathematicians by writing $P(t)$ as an abbreviation for $P(x(t),\,y(t))$ and $Q(t) = Q(x(t),\,y(t))$.

The $P\,dx + Q\,dy$ notation is very common and often convenient. However it looks rather strange because one is not used to seeing a differential that is the sum of two parts. One could make it less strange by instead writing $\int_{\mathcal{C}} P\,dx + \int_{\mathcal{C}} Q\,dy$. Although this is quite correct, and corresponds more closely to the actual process of doing a calculation, it is conceptually less desirable, since it makes the line integral seem like two things rather than a single unified thing.

In practice, in any case, in order to actually calculate $\int_{\mathcal{C}} P(x, y)\,dx$ one usually needs to use a parameter t so that

$$\int_{\mathcal{C}} P(x, y)\,dx = \int_{t_0}^{t_1} P(x(t),\,y(t))\,x'(t)\,dt.$$

In any case, it is important to remember that $\int_{\mathcal{C}} P(x, y)\, dx$ is **not a partial integral**. The variable y in this integral is not an independent parameter, it is a function that depends on the point on the curve, just as x does.

To understand how the line integral $\int_{\mathcal{C}} P\, dx$ is different from an ordinary integral in Calculus I, suppose that $P(x, y) = y$ and $\mathcal{C}$ is the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

The easiest way to do this problem is to parametrize the ellipse as $x(t) = 2\cos t$, $y(t) = 3\sin t$. But it's more enlightening to use a method that exposes us to some possible pitfalls.

First of all, suppose we want to do the integral as written:

$$\oint_{\mathcal{C}} P\, dx = \int_{x_0}^{x_1} y\, dx.$$

Remembering that this is **not a partial integral**, it will be necessary to substitute for y the appropriate function of x. But first there's a more intriguing problem: what should x_0 and x_1 be?

The values x_0 and x_1 should be the x-coordinates of the beginning and end of the curve. But since the ellipse is a closed curve, it starts and ends at the same point. It might make sense, for instance, to think of it starting and ending at $(2, 0)$. Thus $x_0 = x_1 = 2$, so should we write

$$\oint_{\mathcal{C}} y\, dx = \int_{2}^{2} y(x)\, dx\,?$$

But this integral would be 0. And maybe, in fact, 0 is the right answer. But there are a couple of reasons why we should be skeptical of this.

(1) It's always wise to be skeptical of an answer that comes too easily. Sometimes, of course, there really is an extremely easy way to do a problem. But when you get an answer this easily, it's always a good idea to stop and think for a while, to make sure that it really makes sense.

(2) In getting this answer, we never used the formula for P or the formula for $\mathcal{C}$. So the logic of what we've done would say that every line integral around any closed curve would be zero. But that doesn't seem like it ought to be true. If it were true, it would certainly make Green's Theorem completely unnecessary.

Since it doesn't work to take $x_0 = x_1$, we need to rethink the process. One idea would be that since $(-2, 0)$ is the point on the ellipse furthest to the left, and $(2, 0)$ is the point furthest to the right, we should take

$$\oint_C y\, dx = \int_{-2}^{2} y(x)\, dx \ (?)$$

But this also turns out to be incorrect.

In learning mathematics, it's very important to make mistakes. Often only the process of trying to figure out what's wrong when you've made a mistake can cause you to think about what's happening deeply enough to get real insight.

In this case, what's happening is that the line integral has to go all the way around the ellipse. When it does this, x will start at a certain x_0 and end at the same x_0, but in between the point must travel over both the top and the bottom of the ellipse.

Now x alone is inadequate to determine whether one is one the top or bottom of the ellipse. To know this, we have to know y.

We can write

$$\oint_C y\, dx = \int_{2}^{-2} y\, dx + \int_{-2}^{2} y\, dx.$$

This is correct, but misleading because it is overly abbreviated. In this formula, it **seems that** the two formulas are negatives of each other and hence cancel. But this doesn't happen, because **it's not the same y** in the two formulas. The first integral is on the top of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, so that $y = \frac{1}{3}\sqrt{1 - (x^2/4)}$ and the second integral is on the bottom, so $y = -\frac{1}{3}\sqrt{1 - (x^2/4)}$.

Thus we finally get the correct formula for the integral around the ellipse:

$$\oint_C y\, dx = \int_{2}^{-2} \frac{1}{3}\sqrt{1 - \frac{x^2}{4}}\, dx + \int_{-2}^{2} -\frac{1}{3}\sqrt{1 - \frac{x^2}{4}}\, dx$$

$$= -\frac{2}{3} \int_{-2}^{2} \sqrt{1 - \frac{x^2}{4}}\, dx.$$

Proof of Green's Theorem

Suppose now we have a situation similar to the ellipse just discussed, namely a simple closed curve $\mathcal{C}$ consisting of a bottom half, given by the graph of a function $y = \varphi(x)$ for x between x_0 and x_1 and a top half given by $y = \psi(x)$, for x between x_1 and x_0. We assume that this curve is oriented counterclockwise, as will be the case if $x_0 < x_1$. Also continue to suppose that P and Q are functions of x and y having continuous partial derivatives throughout the region enclosed by $\mathcal{C}$.

Then using the logic above, we get

$$\oint_{\mathcal{C}} P\, dx = \int_{x_0}^{x_1} P(x, \varphi(x))\, dx + \int_{x_1}^{x_0} P(x, \psi(x))\, dx$$

$$= \int_{x_0}^{x_1} P(x, \varphi(x))\, dx - \int_{x_0}^{x_1} P(x, \psi(x))\, dx\,.$$

Now let Ω be the area enclosed by $\mathcal{C}$. Using the usual rules for double integrals,

$$\iint_{\Omega} \frac{\partial P}{\partial y}\, dx\, dy = \int_{x_0}^{x_1} \int_{\varphi(x)}^{\psi(x)} \frac{\partial P(x, y)}{\partial y}\, dy\, dx\,.$$

Notice that, as previously discussed, these integrals, unlike the line integral, are **partial integrals**, so when we integral $\dfrac{\partial P}{\partial y}$ with respect to dy we get P:

$$\int_{\varphi(x)}^{\psi(x)} \frac{\partial P}{\partial y}\, dy = P(x, y)\Big|_{y=\varphi(x)}^{y=\psi(x)}$$

$$= P(x, \psi(x)) - P(x, \varphi(x))\,.$$

Thus

$$\iint_{\Omega} \frac{\partial P}{\partial y}\, dx\, dy = \int_{x_0}^{x_1} \int_{\varphi(x)}^{\psi(x)} \frac{\partial P(x, y)}{\partial y}\, dy\, dx$$

$$= \int_{x_0}^{x_1} P(x,\, \psi(x)) - P(x,\, \varphi(x))\, dx\,.$$

On the other hand, as we have seen,

$$\oint_{\mathcal{C}} P(x, y)\, dx = \int_{x_0}^{x_1} P(x,\, \varphi(x))\, dx - \int_{x_0}^{x_1} P(x,\, \psi(x))\, dx\,.$$

It is now obvious that

$$\oint_{\mathcal{C}} P(x, y)\, dx = -\iint_{\Omega} \frac{\partial P}{\partial y}\, dx\, dy\,.$$

Now we have the proof of Green's Theorem. It is worthwhile at this point to go back to the very beginning of the explanation and answer the questions that arose.

It seemed at first that the formula

$$\iint_\Omega \frac{\partial P}{\partial y}\, dy\, dx = -\oint_{\mathcal{C}} P(x,y)\, dx$$

was almost obvious from the fact that when you integrate $\dfrac{\partial P}{\partial y}$ with respect to dy you should get P. However we noticed a few puzzling questions.

(1) Why does this formula have a minus sign in it? (This is the most obvious signal that the "proof" originally given was sliding over some subtleties.)

(2) Why is right-hand side of this formula a line integral?

(3) When we get P by integrating $\dfrac{\partial P}{\partial y}$ with respect to y, why does the answer still have a y in it?

We can now answer these questions.

(2) The right-hand side of the formula in (2) is a line integral over $\mathcal{C}$ because in evaluating the double integral, P has to be integrated between the bottom boundary and the top boundary of Ω and these boundaries are what make up $\mathcal{C}$.

(3) When we integrate $P(x,y)$ with respect to y, the answer only looks like it still has a y in it. When we do the line integral, y is not an independent variable but gets replaced by a function of x.

(1) The minus sign on the right-hand side of the formula occurs because the line integral is done counter-clockwise, and this means that we go backwards over the top of the boundary of Ω and forwards over the bottom, which is exactly the opposite of what is required for the double integral.

Finally, we should note that this proof only applies to a special case. There are many regions Ω where the top and bottom boundaries will not be graphs for functions $y = \psi(x)$ and $y = \varphi(x)$. But this special case basically shows the logic that makes Green's Theorem true. Most calculus books show how to generalize the proof for regions with more complicated boundaries.

We trust that the student will easily figure out for him/herself to prove the other half needed for Green's Theorem, namely

$$\oint_{\mathcal{C}} Q\, dy = \iint_\Omega \frac{\partial Q}{\partial x}\, dx\, dy.$$

Example. Let $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$ and let $\mathcal{C}$ be the circle of radius a centered at the origin. We can parametrize $\mathcal{C}$ in the usual way:

$$\mathbf{r}(\theta) = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j}.$$

We then see that $\mathbf{r}'(\theta) = -a\sin\theta\,\mathbf{i} + a\cos\theta\,\mathbf{j} = -y\,\mathbf{i} + x\,\mathbf{j} = \mathbf{F}(\mathbf{r}(\theta))$, so that

$$\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{dr} = \int_0^{2\pi} \mathbf{F}\cdot\mathbf{r}'\,d\theta = \int_0^{2\pi} \mathbf{F}\cdot\mathbf{F}\,d\theta = a^2\int_0^{2\pi} d\theta = 2\pi a^2.$$

On the other hand,

$$\iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \iint_\Omega 1 - (-1)\,dx\,dy = 2\pi a^2,$$

since the area of Ω is πa^2. Thus the two calculations give the same result, as predicted by Green's Theorem. $\boxed{\checkmark}$

There are still a few comments on Green's Theorem that needed to be added. To start with, an extremely important one:

Green's Theorem is not the formula we have proved.

Instead, Green's Theorem consists of this formula together with the words that surround it. Now it is understandable that one should give slight attention to these words, regarding them as essential but not very important wrapping paper enveloping the assertion of real interest. But to take this point of view is the make a very serious error.

The most important part of the wrapping paper here is the assertion in Green's Theorem that P and Q must be differentiable functions of (x, y) within the area enclosed by $\mathcal{C}$ and their partial derivatives must be continuous. The assertion that $\mathcal{C}$ be a simple closed curve is also important, but there is less of a temptation to get oneself into trouble by forgetting it.

Consider the following example:

$$P = \frac{-y}{x^2 + y^2}, \qquad Q = \frac{x}{x^2 + y^2}.$$

These functions are mildly unpleasant to differentiate, but if one takes the effort one finds that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} = 0.$$

Now let $\mathcal{C}$ be a circle with radius a around the origin. We can parametrize $\mathcal{C}$ in the usual way:

$$\mathbf{r}(\theta) = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j}, \qquad -0 \le \theta \le 2\pi.$$

(We have denoted the parameter by θ rather than t in order to notice a relationship to polar coordinates.)

If we now attempt to calculate $\oint_{\mathcal{C}} P\,dx + Q\,dy$ using Green's Theorem, we get $\iint_{\Omega} 0\,dx\,dy = 0$. However, it is easy to do the integral directly and thus see that the correct answer is different.

In fact, notice that in polar coordinates

$$P = \frac{-y}{x^2 + y^2} = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}, \qquad Q = \frac{\cos\theta}{r},$$

where, as usual, $r = \sqrt{x^2 + y^2}$, so that $r = a$ on the circle $\mathcal{C}$. Furthermore, we see immediately that

$$\mathbf{r}'(\theta) = -a\sin\theta\,\mathbf{i} + a\cos\theta\,\mathbf{j},$$

so that

$$\int_{\mathcal{C}} P\,dx + Q\,dy = \int_{\mathcal{C}} (P\mathbf{i} + Q\mathbf{j})\cdot\mathbf{r}'\,d\theta = \int_0^{2\pi} \frac{a(\sin^2\theta + \cos^2\theta)}{a}\,d\theta = 2\pi.$$

Green's Theorem fails for this example, because $P(x,y)$ and $Q(x,y)$ are not continuous at the origin, where P and Q have zero in the denominator.

It is at first somewhat amazing that the failure of P and Q to be continuous at a single point could cause Green's Theorem to go bad here. But it is indeed the case, and is a little less surprising if one realizes that $\dfrac{\partial Q}{\partial x}$ and $\dfrac{\partial P}{\partial y}$ both blow up as (x,y) approaches the origin, so that $\iint_{\Omega} \dfrac{\partial Q}{\partial x}\,dx\,dy$ and $\iint_{\Omega} \dfrac{\partial P}{\partial y}\,dx\,dy$ become improper integrals and, as far as I know, don't converge.

It is interesting that in the line integral in this example, the result, 2π, is independent of the radius a of the circle. In fact, by doing the calculation slightly differently, we can see that much more is true. Let now $\mathcal{C}$ be any simple closed curve going around the origin counterclockwise and which can be described in polar coordinates: $r = r(\theta)$. This gives a parametrization of the curve with θ as the parameter:

$$x = r(\theta)\cos\theta, \qquad y = r(\theta)\sin\theta.$$

From this, we get by the product rule,

$$dx = \frac{dx}{d\theta}\, d\theta = (r' \cos\theta - r \sin\theta)\, d\theta$$

$$dy = \frac{dy}{d\theta}\, d\theta = (r' \sin\theta + r \cos\theta)\, d\theta.$$

Changing P and Q to polar coordinates as before, we get

$$\int_{\mathcal{C}} P\, dx + Q\, dy = \int_0^{2\pi} \frac{-r\sin\theta(r'\cos\theta - r\sin\theta) + r\cos\theta(r'\sin\theta - r\sin\theta)}{r^2}\, d\theta$$

$$= \int_0^{2\pi} \frac{r^2(\sin^2\theta + \cos^2\theta)}{r^2}\, d\theta = 2\pi.$$

Thus the answer is the same for any closed curve going around the origin counterclockwise. (As the curve goes around the origin, it sweeps through all four quadrants of the plane, so that θ goes from 0 to 2π.)

This fact can actually be seen to be a consequence of Green's Theorem.

I also want to say that it's a little disingenous of me to state that Green's Theorem is actually two separate theorems, one for P and one for Q. Although this is a useful way to think of it for purposes of finding a proof, it really puts too much emphasis on the coordinates x and y and obscures the fact that Green's Theorem has a significance that transcends a particular choice of a coordinate system.

Furthermore, the expression

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

actually has an intrinsic significance for any vector field. It is essentially what in three dimensional vector analysis is called the "curl," and measures the extent to which there is a rotation in the vector field. (The easiest way of understanding this is actually through the use of Green's Theorem.)

For the moment, let's just point out a simple case showing the significance of the curl.

Theorem. If $P(x,y)$ and $Q(x,y)$ are continuously differentiable functions throughout a certain region and if

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

through out that region, then for any simple closed curve $\mathcal{C}$ whose interior is completely enclosed within the region,

$$\oint_{\mathcal{C}} P\, dx + Q\, dy = 0.$$

PROOF: This is an immediate consequence of Green's Theorem. ☑