

Kepler's Second Law says that as a planet moves in its orbit around the sun, it will sweep out equal areas in equal times. That is, the area swept out by the radius vector from the sun to the planet in a given amount of time will always be the same, regardless of where the planet is in its orbit.

It is natural to think that Kepler's Second Law is a consequence of Newton's Inverse-Square Law of Gravitation. But, in fact, as the calculation in Salas & Hille shows, it is simply a consequence of the fact that gravitation is a central force. Since the gravitational force on the planet always pulls it directly towards the sun, the resulting torque, which is given as the cross product of this force with the radius vector, will be zero and therefore the angular momentum is constant. Now it can be seen that that the formula for angular momentum in polar coordinates (with the sun at the origin of the coordinate system, of course) is $L = mr^2\theta'(t)$. Kepler's Second Law then follows from the fact that L is constant, since as is well known the formula for area in polar coordinates is

$$\frac{1}{2} \int_{\theta_0}^{\theta_1} r^2 d\theta = \frac{1}{2} \int_{t_0}^{t_1} r^2 \theta'(t) dt = \frac{(t_1 - t_0)L}{2m}.$$

Now the fact that angular momentum equals $mr^2\theta'(t)$ is really fairly obvious if you look at it in the right way. Angular momentum is mass times the magnitude of the cross product of the radius vector with the velocity vector. But the cross product of the radius vector with the radial component of the velocity will be zero, since these are parallel. On the other hand, the magnitude of the tangential component of the velocity vector is easily seen to be $r\theta'(t)$. Since this tangential component is perpendicular to the radius vector, the magnitude of their cross product will then be simply $r^2\theta'(t)$. That, then, is the magnitude of the cross product of the radius vector with the velocity vector, so that the angular momentum is in fact $mr^2\theta'(t)$, as claimed.

Below, this is spelled out in a more computational format.

Theorem. Suppose that an object of mass m is moving in the plane. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ be the position vector at time t and let $r(t) = \|\mathbf{r}(t)\|$ and let $\theta(t)$ be the angle that \mathbf{r} makes with the x -axis. (In other words, (r, θ) are the polar coordinates of the object). Then the angular momentum is given by $\mathbf{L}(t) = mr^2\theta'(t)\mathbf{k}$.

PROOF: Let $\mathbf{u}(t) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$. Then \mathbf{u} is a unit vector in the direction of \mathbf{r} and $\mathbf{r}(t) = r(t)\mathbf{u}(t)$. Differentiate by the product rule to get $\mathbf{v}(t) = \mathbf{r}'(t) = r'(t)\mathbf{u}(t) + r(t)\mathbf{u}'(t)$. Now $\mathbf{u}'(t) = (-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\theta'(t)$, showing that $\mathbf{u}' \perp \mathbf{u}$ and $\|\mathbf{u}'\| = \theta'$. (You should also be able to see this intuitively.) Consequently, $\mathbf{r} \times \mathbf{u}' = \|\mathbf{r}\| \|\mathbf{u}'\| \mathbf{k} = r\theta' \mathbf{k}$. Since $r'\mathbf{u}$ is parallel to \mathbf{r} , $\mathbf{r} \times r'\mathbf{u} = \mathbf{0}$ and so

$$\mathbf{L} = m\mathbf{r} \times \mathbf{v} = m\mathbf{r} \times (r'\mathbf{u} + r\mathbf{u}') = mr\mathbf{r} \times \mathbf{u}' = mr^2\theta' \mathbf{k}. \quad \square$$

Corollary. If an object with mass m is moving in the plane, then the area swept out by the radius vector between times t_0 and t_1 is

$$A = \frac{1}{2m} \int_{t_0}^{t_1} L(t) dt,$$

where $L(t) = \|\mathbf{L}(t)\|$, i.e. $L(t)$ is the magnitude of the angular momentum vector.

PROOF: By the Theorem, $L(t) = mr^2\theta'$ so that

$$\begin{aligned} A &= \frac{1}{2} \int_{\theta(t_0)}^{\theta(t_1)} r^2 d\theta \\ &= \frac{1}{2} \int_{t_0}^{t_1} r^2\theta'(t) dt \\ &= \frac{1}{2m} \int_{t_0}^{t_1} L(t) dt. \quad \square \end{aligned}$$

Corollary [Kepler/Newton]. If an object moving about the origin in the plane is acted on by (only) a central force, then the radius vector sweeps out equal areas in equal times.

PROOF: Since the force is central, the torque is trivial and so $L(t)$ is a constant L . Therefore the integral in the corollary above reduces to the product of L and $t_1 - t_0$. \square