

Assumptions and Terms. We will consider only functions whose derivatives are continuous at every point where the derivative exists. Although there do exist functions not satisfying this condition, they are very rare and almost never occur in practical applications. We also assume that the critical points for a function are isolated, i. e. that there's not any continuous interval consisting completely of critical points.

A **critical point** is a point where $f'(x) = 0$ or $f'(x)$ does not exist. (Note that discontinuities in f are critical points.)

Basic Principles. (1) Solving a Max-Min problem for a function $f(x)$ amounts to figuring out where $f(x)$ is increasing and where it is decreasing.

- (2) A function can change from increasing to decreasing or vice-versa only at a maximum or minimum or at a discontinuity. At a maximum, the function changes from increasing to decreasing. At a minimum, it changes from decreasing to increasing.
- (3) To see whether $f(x)$ is increasing or decreasing in the interval between two critical points x_1 and x_2 , it is enough either to compare $f(x_1)$ and $f(x_2)$ or to check the sign of $f'(x)$ at any one point in between x_1 and x_2 .
- (4) In any interval where $f(x)$ is increasing and there are no critical points, $f'(x) > 0$. In an interval where $f(x)$ is decreasing and there are no critical points, $f'(x) < 0$.
- (5) At a maximum, the derivative $f'(x)$ is in the process of changing from positive to negative and therefore the derivative $f'(x)$ is **decreasing**. Therefore $f''(x) \leq 0$ (assuming that $f''(x)$ exists at this point).
- (6) At a minimum, the derivative $f'(x)$ is in the process of changing from negative to positive and therefore $f'(x)$ is **increasing**. Therefore $f''(x) \geq 0$.
- (7) At a critical point which is neither a maximum nor a minimum, the function is either increasing on both sides of the critical point or decreasing on both sides. Therefore $f'(x)$ has the same sign on both sides of the critical point, and is either strictly positive or strictly negative. But at the critical point itself, $f'(x) = 0$. Therefore *a critical point which is neither a maximum nor a minimum for $f(x)$ must be either a maximum for $f'(x)$ or a minimum for $f'(x)$. Therefore (assuming that $f''(x)$ exists), $f''(x) = 0$ at a critical point which is neither a maximum nor a minimum.*

From this we see that,

At a point x_0 where $f(x)$ is a maximum.

- (1) $f(x)$ is smaller than $f(x_0)$ for x on either side of x_0 .
- (2) The derivative $f'(x)$ is positive for x to the left of x_0 and negative to the right of x_0 .
- (3) At x_0 , $f'(x)$ is changing from positive to negative, hence $f'(x)$ is **decreasing**.
- (4) At x_0 , $f''(x) \leq 0$.

At a point x_0 where $f(x)$ is a minimum.

- (1) $f(x)$ is larger than $f(x_0)$ for x on either side of x_0 .
- (2) The derivative $f'(x)$ is negative for x to the left of x_0 and positive to the right of x_0 .
- (3) At x_0 , $f'(x)$ is changing from negative to positive, hence $f'(x)$ is **increasing**.
- (4) At x_0 , $f''(x) \geq 0$.

At a critical point x_0 which is neither a maximum nor a minimum.

- (1) $f(x)$ is larger than $f(x_0)$ on one side of x_0 and smaller on the other side.
- (2) $f'(x)$ is either positive on both sides of x_0 or negative on both sides.
- (3) $f'(x)$ has either a minimum or a maximum at x_0 , therefore $f''(x_0) = 0$ (except in the case where $f''(x_0)$ doesn't exist).