Problem. Find $y(x)$ such that $y'' - 2xy' - 2y = 0$.

(This equation is **second-order linear** with non-constant coefficients.)

Power Series Solution. We **assume** that there exists a solution to the D. E. which can be represented by a Taylor series. This is not always the case.

We write

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + a_{k+1} x^{k+1} + a_{k+2} x^{k+2} + \ldots$$

Then

$$\begin{array}{rccccccc}
y = & a_0 & + a_1 x + \cdots & & & + a_k x^k & + a_{k+1} x^{k+1} + \cdots \\
y' = & a_1 & + 2a_2 x + \cdots & + k a_k x^{k-1} & & + (k+1) a_{k+1} x^k & + (k+2) a_{k+2} x^{k+1} + \cdots
\end{array}$$

$$\begin{array}{rcccc}
y'' = & 2a_2 & + 3 \cdot 2 a_3 x + \ldots & + (k+2)(k+1) a_{k+2} x^k & + \cdots \\
-2xy' = & & -2a_1 x - \cdots & -2k a_k x^k & - \cdots \\
-2y = & -2a_0 & -2\, a_1 x + \cdots & -2\, a_k x^k & - \cdots
\end{array}$$

Substituting in the equation $y'' - 2xy' - 2y = 0$ yields

$$\begin{aligned}
2a_2 - 2a_0 &= 0, \\
6a_3 - 4a_1 &= 0, \\
(k+2)(k+1) a_{k+2} - 2(k+1) a_k &= 0.
\end{aligned}$$

Thus $a_{k+2} = 2a_k/(k+2)$. Substituting $r = k-2$ yields $a_r = 2a_{r-2}/r$. Therefore, for instance

$$a_8 = \frac{2a_6}{8} = \frac{2^2 a_4}{8 \cdot 6} = \frac{2^3 a_2}{8 \cdot 6 \cdot 4} = \frac{a_2}{4 \cdot 3 \cdot 2} = \frac{a_0}{4 \cdot 3 \cdot 2}\,,$$

and in general, for any $s \geq 2$,

$$a_{2s} = \frac{2a_{2s-2}}{2s} = \frac{4a_{2s-4}}{4s(s-1)} = \frac{8a_{2s-6}}{8s(s-1)(s-2)} = \cdots = \frac{2^s a_0}{2^s s!} = \frac{a_0}{s!}$$

and for $s \geq 1$,

$$a_{2s+1} = \frac{2a_{2s-1}}{2s+1} = \frac{2^2 a_{2s-3}}{(2s+1)(2s-1)} = \cdots = \frac{2^s a_1}{(2s+1)(2s-1)\cdots 3}\,.$$

Thus

$$y = a_0 \sum_{s=0}^{\infty} \frac{x^{2s}}{s!} + a_1 \sum_{s=0}^{\infty} \frac{2^s x^{2s+1}}{(2s+1)\cdots 3 \cdot 1} \quad = \quad a_0 e^{x^2} + a_1 \sum_{s=0}^{\infty} \frac{2^s x^{2s+1}}{(2s+1)\cdots 3 \cdot 1}$$

where a_0 and a_1 are arbitrary constants.