Sets of equations implying semidistributivity and $n$-permutability

Ralph Freese
Definitions

- $\Sigma$ is a set of equations.
Definitions

- Σ is a set of equations.
- Σ is idempotent: if $F$ is a function symbol occurring in Σ then $Σ \models x \approx F(x, \ldots, x)$. 

A variety $V$ realizes Σ if the function symbols occurring in Σ can be interpreted as $V$-terms such that the equations of Σ hold.

If $x \approx F(w)$, where $w$ is a vector of not necessarily distinct variables, then $F$ is weakly independent of its $i$th place for each $i$ with $w_i \neq x$.

So a Maltsev term $p(x, y, z)$ is weakly independent of all of its places. 

$Σ'$, the derivative is the augmentation of Σ by equations that say that $F$ is independent of its $i$th place whenever $Σ$ implies $F$ is weakly independent of its $i$th place.
Definitions

- $\Sigma$ is a set of equations.
- $\Sigma$ is idempotent: if $F$ is a function symbol occurring in $\Sigma$ then $\Sigma \models x \approx F(x, \ldots, x)$.
- A variety $\mathcal{V}$ realizes $\Sigma$ if the function symbols occurring in $\Sigma$ can be interpreted as $\mathcal{V}$-terms such that the equations of $\Sigma$ hold.
Definitions

- $\Sigma$ is a set of equations.
- $\Sigma$ is idempotent: if $F$ is a function symbol occurring in $\Sigma$ then $\Sigma \models x \approx F(x, \ldots, x)$.
- A variety $\mathcal{V}$ realizes $\Sigma$ if the function symbols occurring in $\Sigma$ can be interpreted as $\mathcal{V}$-terms such that the equations of $\Sigma$ hold.
- If $x \approx F(w)$, where $w$ is a vector of not necessarily distinct variables, then $F$ is weakly independent of its $i^{th}$ place for each $i$ with $w_i \neq x$. 

Ralph Freese ()  
Semidistributivity and $n$-permutability  
Mar 3, 2012 2 / 8
Definitions

- \( \Sigma \) is a set of equations.
- \( \Sigma \) is idempotent: if \( F \) is a function symbol occurring in \( \Sigma \) then \( \Sigma \models x \approx F(x, \ldots, x) \).
- A variety \( V \) realizes \( \Sigma \) if the function symbols occurring in \( \Sigma \) can be interpreted as \( V \)-terms such that the equations of \( \Sigma \) hold.
- If \( x \approx F(w) \), where \( w \) is a vector of not necessarily distinct variables, then \( F \) is weakly independent of its \( i \)-th place for each \( i \) with \( w_i \neq x \). So a Maltsev term \( p(x, y, z) \) is weakly independent of all of its places.
Definitions

- $\Sigma$ is a set of equations.
- $\Sigma$ is idempotent: if $F$ is a function symbol occurring in $\Sigma$ then $\Sigma \models x \approx F(x, \ldots, x)$.
- A variety $\forall$ realizes $\Sigma$ if the function symbols occurring in $\Sigma$ can be interpreted as $\forall$-terms such that the equations of $\Sigma$ hold.
- If $x \approx F(w)$, where $w$ is a vector of not necessarily distinct variables, then $F$ is weakly independent of its $i^{th}$ place for each $i$ with $w_i \neq x$. So a Maltsev term $p(x, y, z)$ is weakly independent of all of its places.
- $\Sigma'$, the derivative is the augmentation of $\Sigma$ by equations that say that $F$ is independent of its $i^{th}$ place whenever $\Sigma$ implies $F$ is weakly independent of its $i^{th}$ place.
Assume throughout this talk that $\Sigma$ is idempotent. Then
The Theorems of Dent, Kearnes, Szendrei

Assume throughout this talk that $\Sigma$ is idempotent. Then

- If $\Sigma'$ is inconsistent then any variety that realizes $\Sigma$ is congruence modular (CM).

If $V$ is a CM variety, then $V$ realizes some $\Sigma$ such that $\Sigma'$ is inconsistent. (The Day terms work.)

The converse of the first statement is false: if $\Sigma$ is the lattice axioms, then $\Sigma' = \Sigma$. But the converse of the first statement is true if $\Sigma$ is linear (no nested composition in the terms occurring in $\Sigma$).

For a finite linear, idempotent $\Sigma$ one can effectively decide if $\Sigma$ implies CM.

This contrasts McNulty's Theorem that there is no effective way to decide if a (nonlinear) idempotent $\Sigma$ implies CM.
Assume throughout this talk that $\Sigma$ is idempotent. Then

- If $\Sigma'$ is inconsistent then any variety that realizes $\Sigma$ is congruence modular (CM).
- If $\mathcal{V}$ is a CM variety, then $\mathcal{V}$ realizes some $\Sigma$ such that $\Sigma'$ is inconsistent. (The Day terms work.)

The converse of the first statement is false: if $\Sigma$ is the lattice axioms, then $\Sigma' = \Sigma$. But the converse of the first statement is true if $\Sigma$ is linear (no nested composition in the terms occurring in $\Sigma$).

For a finite linear, idempotent $\Sigma$ one can effectively decide if $\Sigma$ implies CM. This contrasts McNulty's Theorem that there is no effective way to decide if a (nonlinear) idempotent $\Sigma$ implies CM.
Assume throughout this talk that $\Sigma$ is idempotent. Then

- If $\Sigma'$ is inconsistent then any variety that realizes $\Sigma$ is congruence modular (CM).
- If $\mathcal{V}$ is a CM variety, then $\mathcal{V}$ realizes some $\Sigma$ such that $\Sigma'$ is inconsistent. (The Day terms work.)
- The converse of the first statement is false: if $\Sigma$ is the lattice axioms, then $\Sigma' = \Sigma$. But
Assume throughout this talk that $\Sigma$ is idempotent. Then

- If $\Sigma'$ is inconsistent then any variety that realizes $\Sigma$ is congruence modular (CM).
- If $\mathcal{V}$ is a CM variety, then $\mathcal{V}$ realizes some $\Sigma$ such that $\Sigma'$ is inconsistent. (The Day terms work.)
- The converse of the first statement is false: if $\Sigma$ is the lattice axioms, then $\Sigma' = \Sigma$. But
- The converse of the first statement is true if $\Sigma$ is linear (no nested composition in the terms occurring in $\Sigma$).
Assume throughout this talk that $\Sigma$ is idempotent. Then

- If $\Sigma'$ is inconsistent then any variety that realizes $\Sigma$ is congruence modular (CM).
- If $\mathcal{V}$ is a CM variety, then $\mathcal{V}$ realizes some $\Sigma$ such that $\Sigma'$ is inconsistent. (The Day terms work.)
- The converse of the first statement is false: if $\Sigma$ is the lattice axioms, then $\Sigma' = \Sigma$. But
- The converse of the first statement is true if $\Sigma$ is linear (no nested composition in the terms occurring in $\Sigma$).
- For a finite linear, idempotent $\Sigma$ one can effectively decide if $\Sigma$ implies CM.
Assume throughout this talk that $\Sigma$ is idempotent. Then

- If $\Sigma'$ is inconsistent then any variety that realizes $\Sigma$ is congruence modular (CM).
- If $\mathcal{V}$ is a CM variety, then $\mathcal{V}$ realizes some $\Sigma$ such that $\Sigma'$ is inconsistent. (The Day terms work.)
- The converse of the first statement is false: if $\Sigma$ is the lattice axioms, then $\Sigma' = \Sigma$. But
- The converse of the first statement is true if $\Sigma$ is linear (no nested composition in the terms occurring in $\Sigma$).
- For a finite linear, idempotent $\Sigma$ one can effectively decide if $\Sigma$ implies CM. This contrasts McNulty’s Theorem that there is no effective way to decide if a (nonlinear) idempotent $\Sigma$ implies CM.
A similar theorem holds for $\forall$ satisfying some congruence identity if

"$\Sigma'$ is inconsistent"

is replaced by

"$\Sigma^{(k)}$ is inconsistent for some $k$."

The order derivative, $\Sigma^+$, augments $\Sigma$ by

$$x \approx F(w')$$

whenever $\Sigma \models x \approx F(w)$, where $w'$ is the same as $w$ in every place except one, say $i$, and $w'_i = x$. 
The order derivative, $\Sigma^+$, augments $\Sigma$ by

$$x \approx F(w')$$

whenever $\Sigma \models x \approx F(w)$, where $w'$ is the same as $w$ in every place except one, say $i$, and $w'_i = x$.

If some iterated order derivative $\Sigma^{+k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence $n$-permutable, for some $n$. 
The order derivative, $\Sigma^+$, augments $\Sigma$ by

$$x \approx F(w')$$

whenever $\Sigma \models x \approx F(w)$, where $w'$ is the same as $w$ in every place except one, say $i$, and $w'_i = x$.

If some iterated order derivative $\Sigma^{+k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence $n$-permutabile, for some $n$.

If $\mathcal{V}$ is a congruence $n$-permutabile, for some $n$, then $\mathcal{V}$ realizes some $\Sigma$ whose iterated order derivative $\Sigma^{+k}$ is inconsistent. (The Hagemann-Mitschke terms work.)
The **order derivative**, $\Sigma^+$, augments $\Sigma$ by

$$x \approx F(w')$$

whenever $\Sigma \models x \approx F(w)$, where $w'$ is the same as $w$ in every place except one, say $i$, and $w'_i = x$.

If some iterated order derivative $\Sigma^{+^k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence $n$-permutabile, for some $n$.

If $\mathcal{V}$ is a congruence $n$-permutabile, for some $n$, then $\mathcal{V}$ realizes some $\Sigma$ whose iterated order derivative $\Sigma^{+^k}$ is inconsistent. (The Hagemann-Mitschke terms work.)

The converse of the first statement is false. But
The order derivative, $\Sigma^+$, augments $\Sigma$ by

$$x \approx F(w')$$

whenever $\Sigma \models x \approx F(w)$, where $w'$ is the same as $w$ in every place except one, say $i$, and $w'_i = x$.

If some iterated order derivative $\Sigma^{+k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence $n$-permutabile, for some $n$.

If $\mathcal{V}$ is a congruence $n$-permutabile, for some $n$, then $\mathcal{V}$ realizes some $\Sigma$ whose iterated order derivative $\Sigma^{+k}$ is inconsistent. (The Hagemann-Mitschke terms work.)

The converse of the first statement is false. But

The converse of the first statement is true if $\Sigma$ is linear.
The order derivative, $\Sigma^+$, augments $\Sigma$ by

$$x \approx F(w')$$

whenever $\Sigma \models x \approx F(w)$, where $w'$ is the same as $w$ in every place except one, say $i$, and $w'_i = x$.

If some iterated order derivative $\Sigma^{+k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence $n$-permutabile, for some $n$.

If $\mathcal{V}$ is a congruence $n$-permutabile, for some $n$, then $\mathcal{V}$ realizes some $\Sigma$ whose iterated order derivative $\Sigma^{+k}$ is inconsistent. (The Hagemann-Mitschke terms work.)

The converse of the first statement is false. But

The converse of the first statement is true if $\Sigma$ is linear.

For a finite linear, idempotent $\Sigma$ one can effectively decide if $\Sigma$ implies congruence $n$-permutability, for some $n$. 

Semidistributivity

- The **weak derivative**, $\Sigma^*$, augments $\Sigma$ by an equation expressing that $F$ is independent of its $i^{th}$ place whenever

$$\Sigma \models x \approx F(x, \ldots, x, y, x, \ldots, x)$$

where the $y$ is in the $i^{th}$ place.
The weak derivative, $\Sigma^*$, augments $\Sigma$ by an equation expressing that $F$ is independent of its $i^{th}$ place whenever

$$\Sigma \models x \approx F(x, \ldots, x, y, x, \ldots, x)$$

where the $y$ is in the $i^{th}$ place.

If some iterated weak derivative $\Sigma^*_k$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence semidistributive.
Semidistributivity

- The weak derivative, $\Sigma^*$, augments $\Sigma$ by an equation expressing that $F$ is independent of its $i^{th}$ place whenever

$$\Sigma \models x \approx F(x, \ldots, x, y, x, \ldots, x)$$

where the $y$ is in the $i^{th}$ place.

- If some iterated weak derivative $\Sigma^{*k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence semidistributive.

- If $\mathcal{V}$ is a congruence semidistributive then $\mathcal{V}$ realizes some $\Sigma$ whose iterated weak derivative $\Sigma^{*k}$ is inconsistent. (I’ll show my variant of the Hobby-McKenzie-Kearnes-Kiss terms work.)
• The **weak derivative**, $\Sigma^*$, augments $\Sigma$ by an equation expressing that $F$ is independent of its $i^{th}$ place whenever

$$\Sigma \models x \approx F(x, \ldots, x, y, x, \ldots, x)$$

where the $y$ is in the $i^{th}$ place.

• If some iterated weak derivative $\Sigma^{*k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence semidistributive.

• If $\mathcal{V}$ is a congruence semidistributive then $\mathcal{V}$ realizes some $\Sigma$ whose iterated weak derivative $\Sigma^{*k}$ is inconsistent. (I’ll show my variant of the Hobby-McKenzie-Kearnes-Kiss terms work.)

• The converse of the first statement is false, **even if $\Sigma$ is linear**. Nevertheless
Semidistributivity

- The **weak derivative**, $\Sigma^*$, augments $\Sigma$ by an equation expressing that $F$ is independent of its $i^{th}$ place whenever

$$\Sigma \models x \approx F(x,\ldots,x,y,x,\ldots,x)$$

where the $y$ is in the $i^{th}$ place.

- If some iterated weak derivative $\Sigma^{*^k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence semidistributive.

- If $\mathcal{V}$ is a congruence semidistributive then $\mathcal{V}$ realizes some $\Sigma$ whose iterated weak derivative $\Sigma^{*^k}$ is inconsistent. (I’ll show my variant of the Hobby-McKenzie-Kearnes-Kiss terms work.)

- The converse of the first statement is false, **even if $\Sigma$ is linear**. Nevertheless

- For a finite linear, idempotent $\Sigma$ one **can** effectively decide if $\Sigma$ implies congruence semidistributivity.
A variety is congruence semidistributive iff there are terms $d_i(x, y, z)$, $i = 0, \ldots, n$, such that

$$d_0(x, y, z) \approx x \quad d_n(x, y, z) \approx z$$

and

Let $\Sigma$ be these equations. Assume inductively that $\Sigma^* \implies x \approx d_i(x, y, z)$. Then, using the above equations, one can show that $\Sigma^* + 2 \implies x \approx d_i(x, y, z) + 1$. So some iterated weak derivative implies $x \approx d_n(x, y, z) \approx z$ and so is inconsistent.
A variety is congruence semidistributive iff there are terms \( d_i(x, y, z) \), \( i = 0, \ldots, n \), such that

\[
d_0(x, y, z) \approx x \quad d_n(x, y, z) \approx z
\]

and for each \( i \) two of the following three hold:

\[
\begin{align*}
d_i(x, x, y) &\approx d_{i+1}(x, x, y) \\
d_i(x, y, x) &\approx d_{i+1}(x, y, x) \\
d_i(x, y, y) &\approx d_{i+1}(x, y, y)
\end{align*}
\]
A variety is congruence semidistributive iff there are terms 
\(d_i(x, y, z), \ i = 0, \ldots, n\), such that 

\[ d_0(x, y, z) \approx x \quad d_n(x, y, z) \approx z \]

and for each \(i\) two of the following three hold:

\[ d_i(x, x, y) \approx d_{i+1}(x, x, y) \]

\[ d_i(x, y, x) \approx d_{i+1}(x, y, x) \]

\[ d_i(x, y, y) \approx d_{i+1}(x, y, y) \]

Let \(\Sigma\) be these equations. Assume inductively that \(\Sigma^{*k}\) implies \(x \approx d_i(x, y, z)\). Then, using the above equations, one can show that \(\Sigma^{*k+2}\) implies \(x \approx d_{i+1}(x, y, z)\).
A variety is congruence semidistributive iff there are terms $d_i(x, y, z), \ i = 0, \ldots, n$, such that

$$d_0(x, y, z) \approx x \quad d_n(x, y, z) \approx z$$

and for each $i$ two of the following three hold:

$$d_i(x, x, y) \approx d_{i+1}(x, x, y)$$
$$d_i(x, y, x) \approx d_{i+1}(x, y, x)$$
$$d_i(x, y, y) \approx d_{i+1}(x, y, y)$$

Let $\Sigma$ be these equations. Assume inductively that $\Sigma^{*k}$ implies $x \approx d_i(x, y, z)$. Then, using the above equations, one can show that $\Sigma^{*k+2}$ implies $x \approx d_{i+1}(x, y, z)$.

So some iterated weak derivative implies $x \approx d_n(x, y, z) \approx z$ and so is inconsistent.
Theorem

For each property $P$ listed below, given a finite, idempotent, linear set of equations $\Sigma$ one can effectively decide if every variety that realizes $\Sigma$ satisfies $P$.

- Is congruence modular.
- Satisfies a nontrivial congruence identity.
- Is congruence $n$-permutable for some $n$.
- Is congruence semidistributive.
- Is congruence meet-semidistributive.
- Is congruence distributive.
Theorem

For each property $P$ listed below, given a finite, idempotent, linear set of equations $\Sigma$ one can effectively decide if every variety that realizes $\Sigma$ satisfies $P$.

- Is congruence modular.
- Satisfies a nontrivial congruence identity.
- Is congruence $n$-permutable for some $n$.
- Is congruence semidistributive.
- Is congruence distributive.
Decidable properties of finite, idempotent linear $\Sigma$’s

Theorem

For each property $P$ listed below, given a finite, idempotent, linear set of equations $\Sigma$ one can effectively decide if every variety that realizes $\Sigma$ satisfies $P$.

- Is congruence modular.
- Satisfies a nontrivial congruence identity.
Decidable properties of finite, idempotent linear $\Sigma$’s

Theorem

For each property $P$ listed below, given a finite, idempotent, linear set of equations $\Sigma$ one can effectively decide if every variety that realizes $\Sigma$ satisfies $P$.

- Is congruence modular.
- Satisfies a nontrivial congruence identity.
- Is congruence $n$-permutable for some $n$. 

Ralph Freese ()

Semidistributivity and $n$-permutability

Mar 3, 2012
Decidable properties of finite, idempotent linear $\Sigma$’s

**Theorem**

For each property $P$ listed below, given a finite, idempotent, linear set of equations $\Sigma$ one can effectively decide if every variety that realizes $\Sigma$ satisfies $P$.

- Is congruence modular.
- Satisfies a nontrivial congruence identity.
- Is congruence $n$-permutable for some $n$.
- Is congruence semidistributive.
Decidable properties of finite, idempotent linear $\Sigma$’s

**Theorem**

For each property $P$ listed below, given a finite, idempotent, linear set of equations $\Sigma$, one can effectively decide if every variety that realizes $\Sigma$ satisfies $P$.

- Is congruence modular.
- Satisfies a nontrivial congruence identity.
- Is congruence $n$-permutable for some $n$.
- Is congruence semidistributive.
- Is congruence meet-semidistributive.
Decidable properties of finite, idempotent linear $\Sigma$’s

Theorem

For each property $P$ listed below, given a finite, idempotent, linear set of equations $\Sigma$ one can effectively decide if every variety that realizes $\Sigma$ satisfies $P$.

- Is congruence modular.
- Satisfies a nontrivial congruence identity.
- Is congruence $n$-permutable for some $n$.
- Is congruence semidistributive.
- Is congruence meet-semidistributive.
- Is congruence distributive.