Maltsev Conditions

Ralph Freese and Matthew Valeriote

Jónsson's Conditions for Distributivity

- A variety $\mathcal{V}$ is congruence distributive if and only if there are 3-ary terms $d_0, \ldots, d_k$ (called *Jónsson terms*) such that

$$\begin{aligned} d_0(x, y, z) &\approx x \\ d_i(x, y, x) &\approx x && \text{for } 0 \leq i \leq k \\ d_i(x, x, y) &\approx d_{i+1}(x, x, y) && \text{for all even } i < k \\ d_i(x, y, y) &\approx d_{i+1}(x, y, y) && \text{for all odd } i < k \\ d_k(x, y, z) &\approx z. \end{aligned} \tag{1}$$

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- Let $\mathrm{CD}(k)$ be the class of varieties satisfying (1). Clearly $\mathrm{CD}(k-1) \subseteq \mathrm{CD}(k)$.
- If $\mathcal{V}$ is in $\mathrm{CD}(k)$ but not in $\mathrm{CD}(k-1)$, we say $\mathcal{V}$ has *Jónsson level* k. The Jónsson level of a single algebra $\mathbf{A}$ is the level of $\boldsymbol{V}(\mathbf{A})$.

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- How hard is it to test if $\boldsymbol{V}(\mathbf{A})$ is congruence distributive for a finite $\mathbf{A}$?

A Better Way

$\rho = \rho_1 \cup \rho_3$, the union of the first and third projection kernels on triples.
So

A Better Way

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Theorem

Let $\mathcal{V}$ be a variety and let $\mathbf{S}$ be the subalgebra of $\mathbf{F}^3_{\mathcal{V}}(x, y)$ generated by (x, x, y), (x, y, x) and (y, x, x). Let T be the subset of S consisting of triples whose middle coordinate is x.

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- $\mathcal{V}$ *is congruence distributive iff there is a* ρ*-path in* T *from* (x, x, y) *to* (y, x, x), *where the first link is* ρ_1.
- *If* $\mathcal{V}$ *is congruence distributive then the Jónsson level of* $\mathcal{V}$ *is the length of the shortest such path.*
- *Moreover, if* $\mathcal{V}$ *is congruence distributive then the Jónsson level is at most* $2m - 2$, *where* $m = |\mathbf{F}_{\mathcal{V}}(x, y)|$ *and this is the best possible bound in terms of* m.

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Proof.

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- But there is no polynomial time algorithm. In fact,

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- *does* $\mathbf{A}$ *generate a congruence distributive variety?*
- *does* $\mathbf{A}$ *generate a congruence modular variety?*
- *does* $\mathbf{A}$ *generate a variety that omits all of the types in the set* T*, where* T *is one of:*
  - $\{\mathbf{1}\}$,
  - $\{\mathbf{1}, \mathbf{2}\}$,
  - $\{\mathbf{1}, \mathbf{5}\}$,
  - $\{\mathbf{1}, \mathbf{2}, \mathbf{5}\}$.

Day Quadruples

Let a, b, c and $d \in \mathbf{A}$ and let

$$\alpha = \mathrm{Cg}^{\mathbf{A}}(c, d) \quad \beta = \mathrm{Cg}^{\mathbf{A}}((a, b)(c, d)) \quad \gamma = \mathrm{Cg}^{\mathbf{A}}((a, c)(b, d))$$

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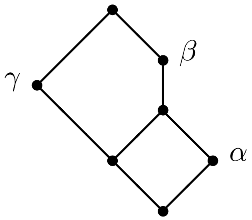

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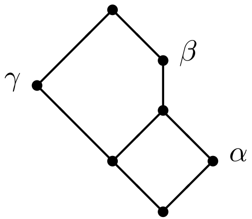

(a, b, c, d) is a **Day quadruple** if in the subalgebra **B** generated by $\{a, b, c, d\}$

$$(a, b) \notin \mathrm{Cg}^{\mathbf{B}}(c, d) \vee [\mathrm{Cg}^{\mathbf{B}}((a, b)(c, d)) \wedge \mathrm{Cg}^{\mathbf{B}}((a, c)(b, d))]$$

Polynomial Algorithms for Idempotent Algebras

Theorem

Let $\mathbf{A}$ *be a finite idempotent algebra and* $\mathcal{V}$ *be the variety it generates. Then* $\mathcal{V}$ *fails to be congruence modular if and only if there is a Day quadruple,* (a, b, c, d) *in* $\mathbf{A}^2$*. Moreover, this Day quadruple can be chosen so that*

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- *if* $\mathcal{V}$ *omits both type* **1** *and type* **5** *then these elements may be chosen so that* $d \in \mathrm{Sg}^{\mathbf{A^2}}(a, b, c)$ *and* $c \in \mathrm{Sg}^{\mathbf{A^2}}(a, b, d)$*.*

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- *there exist* x_0, x_1, y_0, y_1 *in* $\mathbf{A}$ *such that* $a = (x_0, x_1)$, $b = (x_0, y_1)$, $c = (y_0, x_1)$, *and* $d = (y_0, y_1)$;
- *if* $\mathcal{V}$ *omits type* **1** *then these elements may be choosen so that* $d \in \mathrm{Sg}^{\mathbf{A}^2}(a, b, c)$;
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Polynomial Algorithms for Idempotent Algebras

$$n = |\mathbf{A}|$$

$$m = ||\mathbf{A}|| = \sum_{i=0}^{r} k_i n^i$$

$$r = \text{the largest arity of the operations of } \mathbf{A}$$

$(k_i = \text{the number of basic operations of arity } i)$

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$\boldsymbol{V}(\mathbf{A})$ *is congruence modular:* $crn^4 m^2$.

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$\mathbf{A}$ *has a majority term:*	crn^6m^2.

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- A variety $\mathcal{V}$ is congruence distributive if and only if there are 3-ary terms $d_0, \ldots, d_k$ such that

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- Let $\mathrm{CD}'(k)$ be the class of varieties satisfying (2).
- What is the relationship between $\mathrm{CD}(k)$ and $\mathrm{CD}'(k)$?

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- What about even $k > 2$?
- Is $\mathrm{CD}'(2k) \subsetneq \mathrm{CD}(2k)$?

Hagemann-Mitschke Terms for k Permutability

$\mathcal{V}$ has k-permutable congruences if and only if it has ternary terms $p_0, p_1, \ldots, p_k$ such that $p_0(x, y, z) \approx x$, $p_k(x, y, z) \approx z$, and, for $i = 0, \ldots, k-1$,

$$p_i(x, x, y) \approx p_{i+1}(x, y, y).$$

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The minimal such k is at most $|\mathbf{F}_{\mathcal{V}}(x, y)|$.

Kearnes' Example

Let $\mathbf{K}_n = \{0, 1, \ldots, n-1\}$ as a lattice. Let

$$p_i(x) = \begin{cases} i & \text{if } x < i \\ i-1 & \text{otherwise} \end{cases}$$

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Then for $i = 1, \ldots, n-1$,

$$h_i(x, y, z) = (x \wedge z) \vee [x \wedge p_{n-i}(y)] \vee [z \wedge p_i(y)]$$

are Hagemann-Mitschke terms.

The Reduct is to these Terms is Distributive

- Jónsson terms:

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d_i(x, y, z) &= h_i(x, y, z) && i \text{ even} \\
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- $\mathcal{V} \in \mathrm{CD}'(2)$ *if and only if it has a Pixley term.*
- $\mathrm{CD}'(2) \subsetneq \mathrm{CD}(2)$.

The ALVIN Variant Revisited

Proposition

- $\mathrm{CD}(k) \subseteq \mathrm{CD}'(k+1)$ *and* $\mathrm{CD}'(k) \subseteq \mathrm{CD}(k+1)$*.*
- $\mathrm{CD}(k) = \mathrm{CD}'(k)$ *when k is odd.*
- $\mathcal{V} \in \mathrm{CD}(2)$ *if and only if it has a majority term.*
- $\mathcal{V} \in \mathrm{CD}'(2)$ *if and only if it has a Pixley term.*
- $\mathrm{CD}'(2) \subsetneq \mathrm{CD}(2)$*.*
- $\mathrm{CD}'(2k) \not\subseteq \mathrm{CD}(2k)$ *and* $\mathrm{CD}'(2k) \not\subseteq \mathrm{CD}(2k)$ *for* $k > 1$*.*

The End

The End

Trailer ...

The Theorem

Theorem

Let $\mathbf{A}$ *be a finite idempotent algebra and* $\mathcal{V}$ *be the variety it generates. Then* $\mathcal{V}$ *fails to be congruence modular if and only if there is a Day quadruple,* (a, b, c, d) *in* $\mathbf{A}^2$*. Moreover, this Day quadruple can be chosen so that*

- *there exist* x_0, x_1, y_0, y_1 *in* $\mathbf{A}$ *such that* $a = (x_0, x_1)$, $b = (x_0, y_1)$, $c = (y_0, x_1)$, *and* $d = (y_0, y_1)$;
- *if* $\mathcal{V}$ *omits type* $\mathbf{1}$ *then these elements may be choosen so that* $d \in \mathrm{Sg}^{\mathbf{A}^2}(a, b, c)$;
- *if* $\mathcal{V}$ *omits both type* $\mathbf{1}$ *and type* $\mathbf{5}$ *then these elements may be chosen so that* $d \in \mathrm{Sg}^{\mathbf{A}^2}(a, b, c)$ *and* $c \in \mathrm{Sg}^{\mathbf{A}^2}(a, b, d)$.